# Two-component Spinor Formalism: practical methods for treating Majorana fermions

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This lecture is based on work that appears in:

H.K. Dreiner, H.E. Haber and S.P. Martin, "Two-component spinor techniques and Feynman Rules for quantum field theory and supersymmetry," Physics Reports (2010), doi:10.1016/j.physrep.2010.05.002. Also available from arXiv:0812.1594v5 [hep-ph].

A nice treatment of four-component Feynman rules for Majorana fermions can be found in:

A. Denner, H. Eck, O. Hahn and J. Kublbeck, "Compact Feynman rules for Majorana fermions," Phys. Lett. **B291** (1992) 278; "Feynman rules for fermion number violating interactions," Nucl. Phys. **B387** (1992) 467.

# Outline

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# Two-component spinors

First, recall that 4-vectors transform under Lorentz transformations, $\Lambda^\mu{}_\nu$, as $p'^{\,\mu} = \Lambda^\mu{}_\nu p^\nu$, where $\Lambda \in$ SO(3,1) satisfies $\Lambda^\mu{}_\nu g_{\mu\rho} \Lambda^\rho{}_\lambda = g_{\nu\lambda}$.* A Lorentz transformation corresponds to a rotation by $\theta$ about an axis $\hat{\boldsymbol{n}}$ $[\vec{\boldsymbol{\theta}} \equiv \theta\hat{\boldsymbol{n}}]$ and a boost, $\vec{\boldsymbol{\zeta}} = \hat{\boldsymbol{v}} \tanh^{-1} |\vec{\boldsymbol{v}}|$, where $\vec{\boldsymbol{v}}$ is the corresponding velocity. Under the same Lorentz transformation, a generic field transforms as:

$$\Phi'(x') = M_R(\Lambda)\Phi(x)\,,$$

where $M_R \equiv \exp\left(-i\vec{\boldsymbol{\theta}}\boldsymbol{\cdot}\vec{\boldsymbol{J}} - i\vec{\boldsymbol{\zeta}}\boldsymbol{\cdot}\vec{\boldsymbol{K}}\right)$ are $N \times N$ representation matrices of the Lorentz group. Defining $\vec{\boldsymbol{J}}_+ \equiv \frac{1}{2}(\vec{\boldsymbol{J}} + i\vec{\boldsymbol{K}})$ and $\vec{\boldsymbol{J}}_- \equiv \frac{1}{2}(\vec{\boldsymbol{J}} - i\vec{\boldsymbol{K}})$,

$$[J^i_\pm\,,\,J^j_\pm] = i\epsilon^{ijk} J^k_\pm\,, \qquad [J^i_\pm\,,\,J^j_\mp] = 0\,.$$

Thus, the representations are characterized by $(j_1, j_2)$, where the $j_i$ are half-integers. $(0,0)$ is a scalar and $(\frac{1}{2},\frac{1}{2})$ is a four-vector. Of interest to us are the spinor representations $(\frac{1}{2},0)$ and $(0,\frac{1}{2})$.

*In our conventions $g_{\mu\nu} = \mathrm{diag}(1\,,\,-1\,,\,-1\,,\,-1)$.

$(\frac{1}{2},0)$: $\quad M=\exp\left(-\frac{i}{2}\vec{\boldsymbol{\theta}}\boldsymbol{\cdot}\vec{\boldsymbol{\sigma}}-\frac{1}{2}\vec{\boldsymbol{\zeta}}\boldsymbol{\cdot}\vec{\boldsymbol{\sigma}}\right)$, but also $(M^{-1})^T=i\sigma^2 M(i\sigma^2)^{-1}$

$(0,\frac{1}{2})$: $[M^{-1}]^\dagger=\exp\left(-\frac{i}{2}\vec{\boldsymbol{\theta}}\boldsymbol{\cdot}\vec{\boldsymbol{\sigma}}+\frac{1}{2}\vec{\boldsymbol{\zeta}}\boldsymbol{\cdot}\vec{\boldsymbol{\sigma}}\right)$, but also $M^*=i\sigma^2[M^{-1}]^\dagger(i\sigma^2)^{-1}$

since $(i\sigma^2)\vec{\boldsymbol{\sigma}}(i\sigma^2)^{-1}=-\vec{\boldsymbol{\sigma}}^*=-\vec{\boldsymbol{\sigma}}^T$

## Transformation laws of 2-component fields

$$\begin{aligned}
\xi'_\alpha &= M_\alpha{}^\beta\,\xi_\beta\,,\\
\xi'^{\,\alpha} &= [(M^{-1})^T]^\alpha{}_\beta\,\xi^\beta\,,\\
\xi'^{\,\dagger\dot{\alpha}} &= [(M^{-1})^\dagger]^{\dot{\alpha}}{}_{\dot{\beta}}\,\xi^{\dagger\dot{\beta}}\,,\\
\xi'^{\,\dagger}_{\dot{\alpha}} &= [M^*]_{\dot{\alpha}}{}^{\dot{\beta}}\xi^\dagger_{\dot{\beta}}\,.
\end{aligned}$$

We use $i\sigma^2=\left(\begin{smallmatrix}0&1\\-1&0\end{smallmatrix}\right)=\epsilon^{\alpha\beta}=\epsilon^{\dot{\alpha}\dot{\beta}}$ and $(i\sigma^2)^{-1}=-i\sigma^2=\epsilon_{\alpha\beta}=\epsilon_{\dot{\alpha}\dot{\beta}}$ to raise and lower spinor indices: $\xi^\alpha=\epsilon^{\alpha\beta}\,\xi_\beta\,;\;\xi^\dagger_{\dot{\alpha}}=\epsilon_{\dot{\alpha}\dot{\beta}}\,\xi^{\dagger\dot{\beta}}$, etc. Dotted and undotted indices are related by hermitian conjugation: $\xi^\dagger_{\dot{\alpha}}\equiv(\xi_\alpha)^\dagger$.

Finally, we introduce the $\sigma$-matrices:

$$\sigma^\mu_{\alpha\dot{\beta}} = (I_2\,;\,\vec{\boldsymbol{\sigma}})\,, \qquad \overline{\sigma}^{\mu\,\dot{\alpha}\beta} = (I_2\,;\,-\vec{\boldsymbol{\sigma}})\,,$$

where $I_2$ is the $2\times 2$ identity matrix. The spinor index structure derives from the relations:

$$M^\dagger\overline{\sigma}^\mu M = \Lambda^\mu{}_\nu\overline{\sigma}^\nu\,, \qquad M^{-1}\sigma^\mu(M^{-1})^\dagger = \Lambda^\mu{}_\nu\sigma^\nu\,.$$

For example, $(M^\dagger)^{\dot{\alpha}}{}_{\dot{\beta}}\overline{\sigma}^{\dot{\beta}\gamma}M_\gamma{}^\delta = \Lambda^\mu{}_\nu\overline{\sigma}^{\nu\,\dot{\alpha}\delta}$. Note that the matrix $M$ and its inverse have the same spinor index structure. Some useful identities:

$$\sigma^\mu_{\alpha\dot{\alpha}} = \epsilon_{\alpha\beta}\epsilon_{\dot{\alpha}\dot{\beta}}\overline{\sigma}^{\mu\,\dot{\beta}\beta}\,, \qquad \overline{\sigma}^{\mu\,\dot{\alpha}\alpha} = \epsilon^{\alpha\beta}\epsilon^{\dot{\alpha}\dot{\beta}}\sigma^\mu_{\beta\dot{\beta}}\,.$$

The utility of $\sigma^\mu$ is that Lorentz 4-vectors can be built from spinor bilinears:

$$\begin{aligned}\chi'^{\,\alpha}(x')\sigma^\mu_{\alpha\dot{\beta}}\xi'^{\,\dagger\,\dot{\beta}}(x') &= \chi^\alpha(x)[M^{-1}\sigma^\mu(M^{-1})^\dagger]_{\alpha\dot{\beta}}\xi^{\dagger\,\dot{\beta}}(x)\\ &= \Lambda^\mu{}_\nu\,\chi(x)^\alpha\sigma^\nu_{\alpha\dot{\beta}}\xi^{\dagger\,\dot{\beta}}(x)\,.\end{aligned}$$

Spinor indices can be suppressed as long as one adopts a summation convention where we contract indices as follows:

$$ {}^{\alpha}{}_{\alpha} \qquad \text{and} \qquad {}_{\dot\alpha}{}^{\dot\alpha} \,. $$

For example,

$$
\begin{aligned}
\xi\eta &\equiv \xi^\alpha \eta_\alpha , \\
\xi^\dagger \eta^\dagger &\equiv \xi^\dagger_{\dot\alpha} \eta^{\dagger\,\dot\alpha} , \\
\xi^\dagger \overline{\sigma}^\mu \eta &\equiv \xi^\dagger_{\dot\alpha} \overline{\sigma}^{\mu\dot\alpha\beta} \eta_\beta , \\
\xi \sigma^\mu \eta^\dagger &\equiv \xi^\alpha \sigma^\mu_{\alpha\dot\beta} \eta^{\dagger\,\dot\beta} .
\end{aligned}
$$

Note that for anticommuting spinors, e.g.,

$$ \eta\xi \equiv \eta^\alpha \xi_\alpha = -\xi_\alpha \eta^\alpha = +\xi^\alpha \eta_\alpha = \xi\eta \,. $$

It is also useful to note the behavior of spinor products under hermitian conjugation:

$$ (\xi\Sigma\eta)^\dagger = \eta^\dagger \Sigma_r \xi^\dagger \,, \qquad (\xi\Sigma\eta^\dagger)^\dagger = \eta \Sigma_r \xi^\dagger \,, $$

where in each case $\Sigma$ stands for any sequence of alternating $\sigma$ and $\overline{\sigma}$ matrices, and $\Sigma_r$ is obtained from $\Sigma$ by reversing the order of all of the $\sigma$ and $\overline{\sigma}$ matrices.

From the sigma matrices, one can construct the antisymmetrized products:

$$(\sigma^{\mu\nu})_\alpha{}^\beta \equiv \frac{i}{4}\left(\sigma^\mu_{\alpha\dot\gamma}\overline{\sigma}^{\nu\dot\gamma\beta} - \sigma^\nu_{\alpha\dot\gamma}\overline{\sigma}^{\mu\dot\gamma\beta}\right) ,$$

$$(\overline{\sigma}^{\mu\nu})^{\dot\alpha}{}_{\dot\beta} \equiv \frac{i}{4}\left(\overline{\sigma}^{\mu\dot\alpha\gamma}\sigma^\nu_{\gamma\dot\beta} - \overline{\sigma}^{\nu\dot\alpha\gamma}\sigma^\mu_{\gamma\dot\beta}\right) .$$

We may write the $(\frac{1}{2},0)$ and $(0,\frac{1}{2})$ transformation matrices, respectively, as:

$$M = \exp\left(-\tfrac{i}{2}\theta^{\mu\nu}\sigma_{\mu\nu}\right) ,$$

$$(M^{-1})^\dagger = \exp\left(-\tfrac{i}{2}\theta^{\mu\nu}\overline{\sigma}_{\mu\nu}\right) ,$$

where $\theta^{\mu\nu}$ is antisymmetric, with $\theta^{ij} = \epsilon^{ijk}\theta^k$ and $\theta^{i0} = \zeta^i$. Consider a pure boost of an on-shell spinor from its rest frame to the frame where $p^\mu = (E_{\boldsymbol{p}}\,, \vec{\boldsymbol{p}})$, with $E_{\boldsymbol{p}} = (|\vec{\boldsymbol{p}}|^2 + m^2)^{1/2}$. Setting $\theta^{ij} = 0$,

$$M = \exp\left(-\tfrac{1}{2}\vec{\boldsymbol{\zeta}}\boldsymbol{\cdot}\vec{\boldsymbol{\sigma}}\right) = \sqrt{\frac{p\cdot\sigma}{m}} = \frac{E_{\boldsymbol{p}} + m - \vec{\boldsymbol{\sigma}}\boldsymbol{\cdot}\vec{\boldsymbol{p}}}{\sqrt{2m(E_{\boldsymbol{p}} + m)}} ,$$

$$(M^{-1})^\dagger = \exp\left(+\tfrac{1}{2}\vec{\boldsymbol{\zeta}}\boldsymbol{\cdot}\vec{\boldsymbol{\sigma}}\right) = \sqrt{\frac{p\cdot\overline{\sigma}}{m}} = \frac{E_{\boldsymbol{p}} + m + \vec{\boldsymbol{\sigma}}\boldsymbol{\cdot}\vec{\boldsymbol{p}}}{\sqrt{2m(E_{\boldsymbol{p}} + m)}} .$$

## Useful identities and Fierz relations

$$\epsilon_{\alpha\beta}\epsilon^{\gamma\delta} = -\delta_\alpha^\gamma \delta_\beta^\delta + \delta_\alpha^\delta \delta_\beta^\gamma, \qquad \epsilon_{\dot\alpha\dot\beta}\epsilon^{\dot\gamma\dot\delta} = -\delta_{\dot\alpha}^{\dot\gamma} \delta_{\dot\beta}^{\dot\delta} + \delta_{\dot\alpha}^{\dot\delta} \delta_{\dot\beta}^{\dot\gamma},$$

$$\sigma^\mu_{\alpha\dot\alpha}\overline{\sigma}_\mu^{\dot\beta\beta} = 2\delta_\alpha^\beta \delta_{\dot\alpha}^{\dot\beta},$$

$$\sigma^\mu_{\alpha\dot\alpha}\sigma_{\mu\beta\dot\beta} = 2\epsilon_{\alpha\beta}\epsilon_{\dot\alpha\dot\beta}, \qquad \overline{\sigma}^{\mu\dot\alpha\alpha}\overline{\sigma}_\mu^{\dot\beta\beta} = 2\epsilon^{\alpha\beta}\epsilon^{\dot\alpha\dot\beta},$$

$$[\sigma^\mu\overline{\sigma}^\nu + \sigma^\nu\overline{\sigma}^\mu]_\alpha{}^\beta = 2g^{\mu\nu}\delta_\alpha^\beta,$$

$$[\overline{\sigma}^\mu\sigma^\nu + \overline{\sigma}^\nu\sigma^\mu]^{\dot\alpha}{}_{\dot\beta} = 2g^{\mu\nu}\delta_{\dot\beta}^{\dot\alpha},$$

$$\sigma^\mu\overline{\sigma}^\nu\sigma^\rho = g^{\mu\nu}\sigma^\rho - g^{\mu\rho}\sigma^\nu + g^{\nu\rho}\sigma^\mu + i\epsilon^{\mu\nu\rho\kappa}\sigma_\kappa,$$

$$\overline{\sigma}^\mu\sigma^\nu\overline{\sigma}^\rho = g^{\mu\nu}\overline{\sigma}^\rho - g^{\mu\rho}\overline{\sigma}^\nu + g^{\nu\rho}\overline{\sigma}^\mu - i\epsilon^{\mu\nu\rho\kappa}\overline{\sigma}_\kappa.$$

Computations of cross sections and decay rates generally require traces of alternating products of $\sigma$ and $\overline{\sigma}$ matrices:

$$\mathrm{Tr}[\sigma^\mu\overline{\sigma}^\nu] = \mathrm{Tr}[\overline{\sigma}^\mu\sigma^\nu] = 2g^{\mu\nu},$$

$$\mathrm{Tr}[\sigma^\mu\overline{\sigma}^\nu\sigma^\rho\overline{\sigma}^\kappa] = 2\left(g^{\mu\nu}g^{\rho\kappa} - g^{\mu\rho}g^{\nu\kappa} + g^{\mu\kappa}g^{\nu\rho} + i\epsilon^{\mu\nu\rho\kappa}\right),$$

$$\mathrm{Tr}[\overline{\sigma}^\mu\sigma^\nu\overline{\sigma}^\rho\sigma^\kappa] = 2\left(g^{\mu\nu}g^{\rho\kappa} - g^{\mu\rho}g^{\nu\kappa} + g^{\mu\kappa}g^{\nu\rho} - i\epsilon^{\mu\nu\rho\kappa}\right),$$

where $\epsilon^{0123} = -\epsilon_{0123} = +1$ in our conventions. Traces involving an odd number of $\sigma$ and $\overline{\sigma}$ matrices cannot arise, since there is no way to connect the spinor indices consistently.

We shall deal with both commuting and anticommuting spinors, which we shall denote generically by $z_i$. Then, the following identities hold

$$
\begin{aligned}
z_1 z_2 &= -(-1)^A z_2 z_1 \\
z_1^\dagger z_2^\dagger &= -(-1)^A z_2^\dagger z_1^\dagger \\
z_1 \sigma^\mu z_2^\dagger &= (-1)^A z_2^\dagger \overline{\sigma}^\mu z_1 \\
z_1 \sigma^\mu \overline{\sigma}^\nu z_2 &= -(-1)^A z_2 \sigma^\nu \overline{\sigma}^\mu z_1 \\
z_1^\dagger \overline{\sigma}^\mu \sigma^\nu z_2^\dagger &= -(-1)^A z_2^\dagger \overline{\sigma}^\nu \sigma^\mu z_1^\dagger \\
z_1^\dagger \overline{\sigma}^\mu \sigma^\rho \overline{\sigma}^\nu z_2 &= (-1)^A z_2 \sigma^\nu \overline{\sigma}^\rho \sigma^\mu z_1^\dagger \,,
\end{aligned}
$$

where $(-1)^A = +1[-1]$ for commuting [anticommuting] spinors. Finally, the Fierz identities are given by:

$$
\begin{aligned}
(z_1 z_2)(z_3 z_4) &= -(z_1 z_3)(z_4 z_2) - (z_1 z_4)(z_2 z_3) \,, \\
(z_1^\dagger z_2^\dagger)(z_3^\dagger z_4^\dagger) &= -(z_1^\dagger z_3^\dagger)(z_4^\dagger z_2^\dagger) - (z_1^\dagger z_4^\dagger)(z_2^\dagger z_3^\dagger) \,, \\
(z_1 \sigma^\mu z_2^\dagger)(z_3^\dagger \overline{\sigma}_\mu z_4) &= -2(z_1 z_4)(z_2^\dagger z_3^\dagger) \,, \\
(z_1^\dagger \overline{\sigma}^\mu z_2)(z_3^\dagger \overline{\sigma}_\mu z_4) &= \quad 2(z_1^\dagger z_3^\dagger)(z_4 z_2) \,, \\
(z_1 \sigma^\mu z_2^\dagger)(z_3 \sigma_\mu z_4^\dagger) &= \quad 2(z_1 z_3)(z_4^\dagger z_2^\dagger) \,.
\end{aligned}
$$

# Properties of fermion fields

The $(\frac{1}{2},0)$ spinor field $\xi_\alpha(x)$ describes a neutral Majorana fermion. The free-field Lagrangian is:

$$\mathscr{L} = i\xi^\dagger\overline{\sigma}^\mu\partial_\mu\xi - \tfrac{1}{2}m(\xi\xi + \xi^\dagger\xi^\dagger)\,.$$

On-shell, $\xi$ satisfies the free-field Dirac equation, $i\overline{\sigma}^{\mu\dot{\alpha}\beta}\partial_\mu\xi_\beta = m\xi^{\dagger\,\dot{\alpha}}$. The solution is:

$$\xi_\alpha(x) = \sum_s \int \frac{d^3\vec{\boldsymbol{p}}}{(2\pi)^{3/2}(2E_{\boldsymbol{p}})^{1/2}}\left[x_\alpha(\vec{\boldsymbol{p}},s)a(\vec{\boldsymbol{p}},s)e^{-ip\cdot x} + y_\alpha(\vec{\boldsymbol{p}},s)a^\dagger(\vec{\boldsymbol{p}},s)e^{ip\cdot x}\right]\,,$$

and $\xi^\dagger_{\dot{\alpha}} = (\xi_\alpha)^\dagger$. The two-component fermion wave functions, $x$ and $y$ are *commuting* spinors that satisfy the momentum-space Dirac equation:

$$(p\cdot\overline{\sigma})^{\dot{\alpha}\beta}x_\beta = my^{\dagger\,\dot{\alpha}}\,, \qquad (p\cdot\sigma)_{\alpha\dot{\beta}}y^{\dagger\,\dot{\beta}} = mx_\alpha\,,$$

$$(p\cdot\sigma)_{\alpha\dot{\beta}}x^{\dagger\,\dot{\beta}} = -my_\alpha\,, \qquad (p\cdot\overline{\sigma})^{\dot{\alpha}\beta}y_\beta = -mx^{\dagger\,\dot{\alpha}}\,.$$

The spin or helicity is labeled by $s = \pm\frac{1}{2}$. For spin, we quantize in the rest frame along a fixed axis $\hat{\boldsymbol{s}} \equiv (\sin\theta\cos\phi\,,\, \sin\theta\sin\phi\,,\, \cos\theta)$. Eigenstates of $\frac{1}{2}\vec{\boldsymbol{\sigma}}\boldsymbol{\cdot}\hat{\boldsymbol{s}}$ are denoted by $\chi_s$, *i.e.*, $\frac{1}{2}\vec{\boldsymbol{\sigma}}\boldsymbol{\cdot}\hat{\boldsymbol{s}}\,\chi_s = s\chi_s$,. Explicitly,

$$\chi_{1/2}(\hat{\boldsymbol{s}}) = \begin{pmatrix} e^{-i\phi/2}\cos(\theta/2) \\ e^{i\phi/2}\sin(\theta/2) \end{pmatrix}, \quad \chi_{-1/2}(\hat{\boldsymbol{s}}) = \begin{pmatrix} -e^{-i\phi/2}\sin(\theta/2) \\ e^{i\phi/2}\cos(\theta/2) \end{pmatrix}.$$

Introduce the spin 4-vector for massive fermions. For fixed-axis spin states, $S^\mu \equiv (0\,;\,\hat{\boldsymbol{s}})$ in the rest frame, boosting to the frame where $p = (E_{\boldsymbol{p}}\,;\,\vec{\boldsymbol{p}})$,

$$S^\mu = \left(\frac{\vec{\boldsymbol{p}}\boldsymbol{\cdot}\hat{\boldsymbol{s}}}{m}\,;\,\hat{\boldsymbol{s}} + \frac{(\vec{\boldsymbol{p}}\boldsymbol{\cdot}\hat{\boldsymbol{s}})\,\vec{\boldsymbol{p}}}{m(E+m)}\right).$$

Helicity states are defined to be eigenstates of $\frac{1}{2}\vec{\boldsymbol{\sigma}}\boldsymbol{\cdot}\hat{\boldsymbol{p}}$, *i.e.*, $\frac{1}{2}\vec{\boldsymbol{\sigma}}\boldsymbol{\cdot}\hat{\boldsymbol{p}}\,\chi_\lambda = \lambda\chi_\lambda$ $(\lambda = \pm\frac{1}{2})$. The explicit forms for $\chi_\lambda$ are the same as above, with $\theta$ and $\phi$ the polar and azimuthal angles of $\hat{\boldsymbol{p}}$. The spin 4-vector is defined by taking $\hat{\boldsymbol{s}} = \hat{\boldsymbol{p}}$. Thus, $S^\mu = \frac{1}{m}\,(|\vec{\boldsymbol{p}}|\,;\,E\hat{\boldsymbol{p}})$. In the high energy limit, $S^\mu = p^\mu/m + \mathcal{O}(m/E)$.

## Explicit construction of the $x$ and $y$ wave functions

The Dirac equation implies that in the rest frame $x_1 = y^{\dagger\,1}$ and $x_2 = y^{\dagger\,2}$. That is, $x_\alpha(\vec{\boldsymbol{p}}=0) = y^{\dagger\,\dot\alpha}(\vec{\boldsymbol{p}}=0)$ are linear combinations of the $\chi_s$ ($s=\pm\frac{1}{2}$). Choose $x_\alpha(\vec{\boldsymbol{p}}=0,s) = y^{\dagger\,\dot\alpha}(\vec{\boldsymbol{p}}=0,s) = \sqrt{m}\chi_s$, and boost to $\vec{\boldsymbol{p}}\neq 0$:

$$x_\alpha(\vec{\boldsymbol{p}},s) = \sqrt{p\cdot\sigma}\,\chi_s\,, \qquad y_\alpha(\vec{\boldsymbol{p}},s) = 2s\sqrt{p\cdot\sigma}\,\chi_{-s}\,,$$
$$x^{\dagger\,\dot\alpha}(\vec{\boldsymbol{p}},s) = -2s\sqrt{p\cdot\overline{\sigma}}\,\chi_{-s}\,, \qquad y^{\dagger\,\dot\alpha}(\vec{\boldsymbol{p}},s) = \sqrt{p\cdot\overline{\sigma}}\,\chi_s\,.$$

For helicity spinors, replace $s$ with $\lambda$. For massless fermions, we must use helicity spinors. Putting $E=|\vec{\boldsymbol{p}}|$ and $m=0$,

$$x_\alpha(\vec{\boldsymbol{p}},\lambda) = \sqrt{E/2}\,\,(1-2\lambda)\,\chi_\lambda\,,$$
$$y_\alpha(\vec{\boldsymbol{p}},\lambda) = \sqrt{E/2}\,\,(1+2\lambda)\,\chi_{-\lambda}\,,$$
$$x^{\dagger\,\dot\alpha}(\vec{\boldsymbol{p}},\lambda) = \sqrt{E/2}\,\,(1-2\lambda)\,\chi_{-\lambda}\,,$$
$$y^{\dagger\,\dot\alpha}(\vec{\boldsymbol{p}},\lambda) = \sqrt{E/2}\,\,(1+2\lambda)\,\chi_\lambda\,.$$

For a given $\lambda$, only one helicity component of $x$ and $y$ survives.

# Projection operators

$$x_\alpha(\vec{\boldsymbol{p}}, s)x^\dagger_{\dot\beta}(\vec{\boldsymbol{p}}, s) = \tfrac{1}{2}(p_\mu - 2smS_\mu)\sigma^\mu_{\alpha\dot\beta}\,,$$

$$y^{\dagger\,\dot\alpha}(\vec{\boldsymbol{p}}, s)y^\beta(\vec{\boldsymbol{p}}, s) = \tfrac{1}{2}(p^\mu + 2smS^\mu)\overline{\sigma}^{\dot\alpha\beta}_\mu\,,$$

$$x_\alpha(\vec{\boldsymbol{p}}, s)y^\beta(\vec{\boldsymbol{p}}, s) = \tfrac{1}{2}\left(m\delta_\alpha{}^\beta - 2s[S\!\cdot\!\sigma\, p\!\cdot\!\overline{\sigma}]_\alpha{}^\beta\right)\,,$$

$$y^{\dagger\,\dot\alpha}(\vec{\boldsymbol{p}}, s)x^\dagger_{\dot\beta}(\vec{\boldsymbol{p}}, s) = \tfrac{1}{2}\left(m\delta^{\dot\alpha}{}_{\dot\beta} + 2s[S\!\cdot\!\overline{\sigma}\, p\!\cdot\!\sigma]^{\dot\alpha}{}_{\dot\beta}\right)\,.$$

For massless spinors, the helicity projection operators are:

$$x_\alpha(\vec{\boldsymbol{p}}, \lambda)x^\dagger_{\dot\beta}(\vec{\boldsymbol{p}}, \lambda) = (\tfrac{1}{2} - \lambda)p\!\cdot\!\sigma_{\alpha\dot\beta}\,,$$

$$y^{\dagger\,\dot\alpha}(\vec{\boldsymbol{p}}, \lambda)y^\beta(\vec{\boldsymbol{p}}, \lambda) = (\tfrac{1}{2} + \lambda)p\!\cdot\!\overline{\sigma}^{\dot\alpha\beta}\,,$$

$$x_\alpha(\vec{\boldsymbol{p}}, \lambda)y^\beta(\vec{\boldsymbol{p}}, \lambda) = y^{\dagger\,\dot\alpha}(\vec{\boldsymbol{p}}, \lambda)x^\dagger_{\dot\beta}(\vec{\boldsymbol{p}}, \lambda) = 0\,.$$

Summing over $s$ (or $\lambda$) yields:

$$\sum_s x_\alpha(\vec{\boldsymbol{p}}, s)x^\dagger_{\dot\beta}(\vec{\boldsymbol{p}}, s) = p\!\cdot\!\sigma_{\alpha\dot\beta}\,, \qquad \sum_s y^{\dagger\,\dot\alpha}(\vec{\boldsymbol{p}}, s)y^\beta(\vec{\boldsymbol{p}}, s) = p\!\cdot\!\overline{\sigma}^{\dot\alpha\beta}\,,$$

$$\sum_s x_\alpha(\vec{\boldsymbol{p}}, s)y^\beta(\vec{\boldsymbol{p}}, s) = m\delta_\alpha{}^\beta\,, \qquad \sum_s y^{\dagger\,\dot\alpha}(\vec{\boldsymbol{p}}, s)x^\dagger_{\dot\beta}(\vec{\boldsymbol{p}}, s) = m\delta^{\dot\alpha}{}_{\dot\beta}\,.$$

# Covariant spin vectors and the Bouchiat-Michel formulae

Introduce the covariant spin vectors,

$$S^{a\mu} = \left( \frac{\vec{\boldsymbol{p}}\cdot\hat{\boldsymbol{s}}^{\boldsymbol{a}}}{m}\,;\;\hat{\boldsymbol{s}}^{\boldsymbol{a}} + \frac{(\vec{\boldsymbol{p}}\cdot\hat{\boldsymbol{s}}^{\boldsymbol{a}})\,\vec{\boldsymbol{p}}}{m(E+m)} \right)\,, \qquad a = 1, 2, 3\,,$$

where the unit vectors $\hat{\boldsymbol{s}}^{\boldsymbol{a}}$ are mutually orthogonal. For fixed-axis spin states, where $\hat{\boldsymbol{s}}$ denotes the axis direction, we identify $\hat{\boldsymbol{s}}^{\boldsymbol{3}} = \hat{\boldsymbol{s}}$. For helicity states, we identify $\hat{\boldsymbol{s}}^{\boldsymbol{a}} = \hat{\boldsymbol{p}}^{\boldsymbol{a}}$, where the $\hat{\boldsymbol{p}}^{\boldsymbol{a}}$ are an orthonormal triad of unit three-vectors with $\hat{\boldsymbol{p}}^{\boldsymbol{3}} \equiv \hat{\boldsymbol{p}}$. In the latter case, we have the explicit representation,

$$S^{1\mu} = (0\,;\,\hat{\boldsymbol{p}}^{\boldsymbol{1}})\,, \qquad S^{2\mu} = (0\,;\,\hat{\boldsymbol{p}}^{\boldsymbol{2}})\,, \qquad S^{3\mu} = \left( \frac{|\vec{\boldsymbol{p}}\,|}{m}\,;\,\frac{E}{m}\,\hat{\boldsymbol{p}} \right)\,,$$

in a coordinate system where $p^\mu = (E\,;\,\vec{\boldsymbol{p}})$ and $\hat{\boldsymbol{p}} = (\sin\theta\cos\phi,\ \sin\theta\sin\phi,\ \cos\theta)$.

We also define a matrix-valued spin four-vector $\mathcal{S}^\mu$, whose matrix elements are given by:

$$\mathcal{S}^\mu_{ss'} \equiv S^{a\mu}\tau^a_{ss'}\,, \qquad s, s' = \pm\tfrac{1}{2}\,,$$

where $\tau^a_{ss'}$ are the matrix elements of the Pauli matrices.

The Bouchiat-Michel formulae (adapted to two-component spinor notation) are given by:

$$
\begin{aligned}
x_\alpha(\vec{\boldsymbol{p}}, s')x^\dagger_{\dot\beta}(\vec{\boldsymbol{p}}, s) &= \tfrac{1}{2}(p\,\delta_{ss'} - m\mathcal{S}_{ss'})\cdot\sigma_{\alpha\dot\beta}\,, \\
y^{\dagger\dot\alpha}(\vec{\boldsymbol{p}}, s')y^\beta(\vec{\boldsymbol{p}}, s) &= \tfrac{1}{2}(p\,\delta_{ss'} + m\mathcal{S}_{ss'})\cdot\overline{\sigma}^{\dot\alpha\beta}\,, \\
x_\alpha(\vec{\boldsymbol{p}}, s')y^\beta(\vec{\boldsymbol{p}}, s) &= \tfrac{1}{2}\left[m\delta_{ss'}\delta_\alpha{}^\beta - [(\sigma\cdot\mathcal{S}_{ss'})\,(\overline{\sigma}\cdot p)]_\alpha{}^\beta\right]\,, \\
y^{\dagger\dot\alpha}(\vec{\boldsymbol{p}}, s')x^\dagger_{\dot\beta}(\vec{\boldsymbol{p}}, s) &= \tfrac{1}{2}\left[m\delta_{ss'}\delta^{\dot\alpha}{}_{\dot\beta} + [(\overline{\sigma}\cdot\mathcal{S}_{ss'})\,(\sigma\cdot p)]^{\dot\alpha}{}_{\dot\beta}\right]\,.
\end{aligned}
$$

Applying the above results to the helicity spinors in the massless limit,

$$
\begin{aligned}
x_\alpha(\vec{\boldsymbol{p}}, \lambda')x^\dagger_{\dot\beta}(\vec{\boldsymbol{p}}, \lambda) &= (\tfrac{1}{2} - \lambda)\,\delta_{\lambda\lambda'}\,p\cdot\sigma_{\alpha\dot\beta}\,, \\
y^{\dagger\dot\alpha}(\vec{\boldsymbol{p}}, \lambda')y^\beta(\vec{\boldsymbol{p}}, \lambda) &= (\tfrac{1}{2} + \lambda)\,\delta_{\lambda\lambda'}\,p\cdot\overline{\sigma}^{\dot\alpha\beta}\,, \\
x_\alpha(\vec{\boldsymbol{p}}, \lambda')y^\beta(\vec{\boldsymbol{p}}, \lambda) &= -(\tfrac{1}{2} - \lambda')(\tfrac{1}{2} + \lambda)\,[(\sigma\cdot S_-)(\overline{\sigma}\cdot p)]_\alpha{}^\beta\,, \\
y^{\dagger\dot\alpha}(\vec{\boldsymbol{p}}, \lambda')x^\dagger_{\dot\beta}(\vec{\boldsymbol{p}}, \lambda) &= (\tfrac{1}{2} + \lambda')(\tfrac{1}{2} - \lambda)\,[(\overline{\sigma}\cdot S_+)(\sigma\cdot p)]^{\dot\alpha}{}_{\dot\beta}\,,
\end{aligned}
$$

where

$$
\sigma\cdot S_- = \begin{pmatrix} \tfrac{1}{2}\sin\theta & e^{-i\phi}\sin^2\dfrac{\theta}{2} \\ -e^{i\phi}\cos^2\dfrac{\theta}{2} & -\tfrac{1}{2}\sin\theta \end{pmatrix}\,, \qquad \overline{\sigma}\cdot S_+ = \begin{pmatrix} \tfrac{1}{2}\sin\theta & -e^{-i\phi}\cos^2\dfrac{\theta}{2} \\ e^{i\phi}\sin^2\dfrac{\theta}{2} & -\tfrac{1}{2}\sin\theta \end{pmatrix}\,.
$$

# Fermion mass diagonalization

The Lagrangian of a collection of free anti-commuting spin-1/2 "interaction-eigenstate" fields $\hat{\xi}_{\alpha i}(x)$, labeled by flavor index $i$:

$$\mathscr{L} = i\hat{\xi}^{\dagger\, i}\overline{\sigma}^{\mu}\partial_{\mu}\hat{\xi}_i - \tfrac{1}{2}M^{ij}\hat{\xi}_i\hat{\xi}_j - \tfrac{1}{2}M_{ij}\hat{\xi}^{\dagger,i}\hat{\xi}^{\dagger,j}\,,$$

where $M_{ij} \equiv (M^{ij})^*$ is a *complex symmetric* matrix. We shall rewrite this in terms of mass-eigenstate fields $\xi(x) = \Omega^{-1}\hat{\xi}(x)$, where $\Omega$ is unitary and chosen such that

$$\Omega^T M\,\Omega = m = \mathrm{diag}(m_1, m_2, \ldots).$$

In linear algebra, this is called the Takagi-diagonalization of a complex symmetric matrix $M$. To compute the values of the diagonal elements of $m$, one may simply note that

$$\Omega^T M M^\dagger \Omega^* = m^2.$$

$MM^\dagger$ is hermitian, and thus it can be diagonalized by a unitary matrix. Thus, the $m_i$ of the Takagi diagonalization are the non-negative square-roots of the eigenvalues of $MM^\dagger$. In terms of the mass eigenstates,

$$\mathscr{L} = i\xi^{\dagger\, i}\overline{\sigma}^{\mu}\partial_{\mu}\xi_i - \tfrac{1}{2}m_i(\xi_i\xi_i + \xi^{\dagger\, i}\xi^{\dagger\, i})\,.$$

# The Dirac fermion

A charged fermion has twice the number of degrees of freedom as the neutral fermion. If $\chi$ and $\eta$ are oppositely charged and degenerate in mass, then the corresponding free-field Lagrangian is:

$$\mathscr{L} = i\chi^\dagger\overline{\sigma}^\mu\partial_\mu\chi + i\eta^\dagger\overline{\sigma}^\mu\partial_\mu\eta - m(\chi\eta + \chi^\dagger\eta^\dagger)\,.$$

Together, $\chi$ and $\eta^\dagger$ constitute a single Dirac fermion. The corresponding mass matrix is $\left(\begin{smallmatrix} 0 & m \\ m & 0 \end{smallmatrix}\right)$. One could Takagi-diagonalize this matrix, although the corresponding mass eigenstates would not be eigenstates of charge.

The solutions to the corresponding Dirac field equations are:

$$\chi_\alpha(x) = \sum_s \int \frac{d^3\vec{\boldsymbol{p}}}{(2\pi)^{3/2}(2E_{\boldsymbol{p}})^{1/2}}\left[x_\alpha(\vec{\boldsymbol{p}},s)a(\vec{\boldsymbol{p}},s)e^{-ip\cdot x} + y_\alpha(\vec{\boldsymbol{p}},s)b^\dagger(\vec{\boldsymbol{p}},s)e^{ip\cdot x}\right]\,,$$

$$\eta_\alpha(x) = \sum_s \int \frac{d^3\vec{\boldsymbol{p}}}{(2\pi)^{3/2}(2E_{\boldsymbol{p}})^{1/2}}\left[x_\alpha(\vec{\boldsymbol{p}},s)b(\vec{\boldsymbol{p}},s)e^{-ip\cdot x} + y_\alpha(\vec{\boldsymbol{p}},s)a^\dagger(\vec{\boldsymbol{p}},s)e^{ip\cdot x}\right]\,.$$

More generally, for a collection of interaction-eigenstate charged fermion pairs $\hat{\chi}_{\alpha i}(x)$, $\hat{\eta}^i_\alpha(x)$, the free-field Lagrangian is:

$$\mathscr{L} = i\hat{\chi}^{\dagger\, i}\overline{\sigma}^\mu\partial_\mu\hat{\chi}_i + i\hat{\eta}^\dagger_i\overline{\sigma}^\mu\partial_\mu\hat{\eta}^i - M^i{}_j\hat{\chi}_i\hat{\eta}^j - M_i{}^j\hat{\chi}^{\dagger\, i}\hat{\eta}^\dagger_j\,,$$

where $M^i{}_j$ is an arbitrary complex matrix, and $M_i{}^j \equiv (M^i{}_j)^*$. We diagonalize the mass matrix by introducing mass-eigenstates $\chi(x) = L^{-1}\hat{\chi}(x)$ and $\eta(x) = R^{-1}\hat{\eta}(x)$ where $L$ and $R$ are unitary matrices that are chosen such that:

$$L^T M R = m = \mathrm{diag}(m_1, m_2, \ldots),$$

with the $m_i$ real and non-negative. This is the singular-value decomposition of linear algebra, which states that for any complex matrix $M$, the unitary matrices $L$ and $R$ above exist. Due to

$$L^T(MM^\dagger)L^* \;=\; R^\dagger(M^\dagger M)R \;=\; m^2\,,$$

the $m_i$ are the non-negative square roots of the eigenvalues of $MM^\dagger$ (or equivalently, $M^\dagger M$). In terms of the mass eigenstates,

$$\mathscr{L} = i\chi^{\dagger\, i}\overline{\sigma}^\mu\partial_\mu\chi_i + i\eta^\dagger_i\overline{\sigma}^\mu\partial_\mu\eta^i - m_i(\chi_i\eta^i + \chi^{\dagger\, i}\eta^\dagger_i)\,.$$

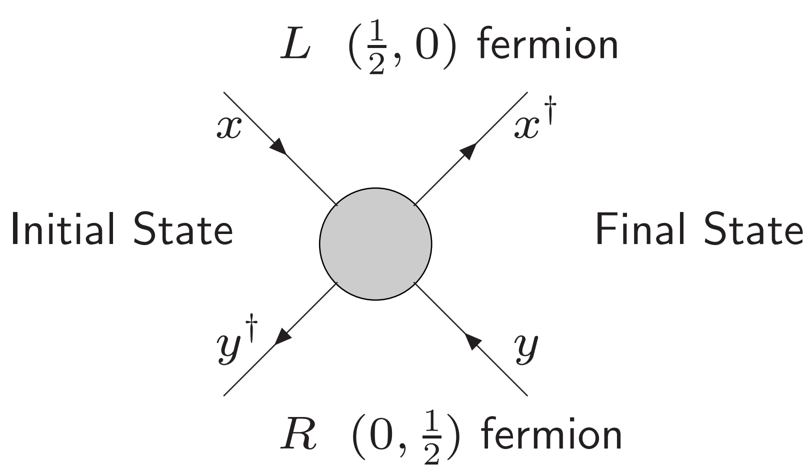
The mass matrix now consists of $2 \times 2$ blocks $\left(\begin{smallmatrix} 0 & m_i \\ m_i & 0 \end{smallmatrix}\right)$ along the diagonal.

# Feynman rules for two-component fermions

The rules for assigning two-component external state spinors are then as follows.

- For an initial-state left-handed $(\frac{1}{2}, 0)$ fermion: $x$.
- For an initial-state right-handed $(0, \frac{1}{2})$ fermion: $y^\dagger$.
- For a final-state left-handed $(\frac{1}{2}, 0)$ fermion: $x^\dagger$.
- For a final-state right-handed $(0, \frac{1}{2})$ fermion: $y$.

The two-component external state fermion wave functions are distinguished by their Lorentz group transformation properties, rather than by their particle or antiparticle status as in four-component Feynman rules. These rules are summarized in the mnemonic diagram:



# Propagators

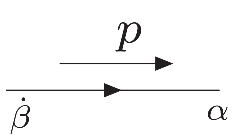
$$\langle 0|\, T\xi_\alpha(x)\bar{\xi}_{\dot\beta}(y)\,|0\rangle_{\rm FT} = \frac{i}{p^2-m^2+i\epsilon}\sum_s x_\alpha(\vec{\boldsymbol{p}},s)\bar{x}_{\dot\beta}(\vec{\boldsymbol{p}},s)$$

$$\langle 0|\, T\bar{\xi}^{\dot\alpha}(x)\xi^{\beta}(y)\,|0\rangle_{\rm FT} = \frac{i}{p^2-m^2+i\epsilon}\sum_s \bar{y}^{\dot\alpha}(\vec{\boldsymbol{p}},s)y^{\beta}(\vec{\boldsymbol{p}},s)$$

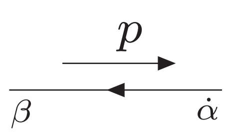
$$\langle 0|\, T\bar{\xi}^{\dot\alpha}(x)\bar{\xi}_{\dot\beta}(y)\,|0\rangle_{\rm FT} = \frac{i}{p^2-m^2+i\epsilon}\sum_s \bar{y}^{\dot\alpha}(\vec{\boldsymbol{p}},s)\bar{x}_{\dot\beta}(\vec{\boldsymbol{p}},s)$$

$$\langle 0|\, T\xi_\alpha(x)\xi^{\beta}(y)\,|0\rangle_{\rm FT} = \frac{i}{p^2-m^2+i\epsilon}\sum_s x_\alpha(\vec{\boldsymbol{p}},s)y^{\beta}(\vec{\boldsymbol{p}},s)$$

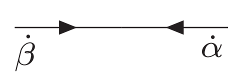
where FT indicates the Fourier transform from position to momentum space. These results have an obvious diagrammatic representation:

(a) 


$$\frac{ip\cdot\sigma_{\alpha\dot\beta}}{p^2-m^2}$$

(b) 


$$\frac{ip\cdot\overline{\sigma}^{\dot\alpha\beta}}{p^2-m^2}$$

(c) 

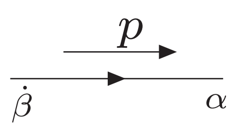

$$\frac{im}{p^2-m^2}\delta^{\dot\alpha}{}_{\dot\beta}$$

(d) 


$$\frac{im}{p^2-m^2}\delta_\alpha{}^\beta$$

Arrows on fermion lines always run away from dotted indices at a vertex and toward undotted indices at a vertex.

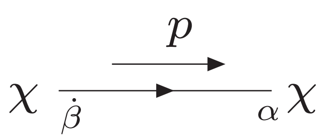
The arrow-preserving propagators can be described by one diagram:



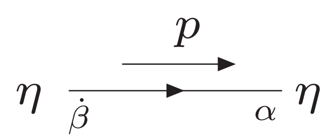

$$\frac{ip\cdot\sigma_{\alpha\dot\beta}}{p^2-m^2} \quad \underline{\text{or}} \quad \frac{-ip\cdot\overline{\sigma}^{\dot\beta\alpha}}{p^2-m^2}$$

Here the choice of the $\sigma$ or the $\overline{\sigma}$ version of the rule is uniquely determined by the height of the indices on the vertex to which the propagator is connected.

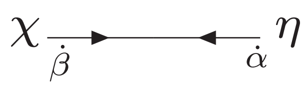
For the case of charged fermions, we write down the rules for propagators involving the charged pair $\chi$ and $\eta$:



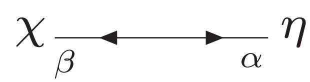

$$\frac{ip\cdot\sigma_{\alpha\dot\beta}}{p^2-m^2} \quad \underline{\text{or}} \quad \frac{-ip\cdot\overline{\sigma}^{\dot\beta\alpha}}{p^2-m^2}$$




$$\frac{ip\cdot\sigma_{\alpha\dot\beta}}{p^2-m^2} \quad \underline{\text{or}} \quad \frac{-ip\cdot\overline{\sigma}^{\dot\beta\alpha}}{p^2-m^2}$$




$$\frac{im}{p^2-m^2}\delta^{\dot\alpha}{}_{\dot\beta}$$




$$\frac{im}{p^2-m^2}\delta_{\alpha}{}^{\beta}$$

# Fermion–scalar interactions

The most general set of interactions with the scalars of the theory $\hat{\phi}_I$ are then given by:

$$\mathscr{L}_{\rm int} = -\tfrac{1}{2}\hat{Y}^{Ijk}\hat{\phi}_I\hat{\psi}_j\hat{\psi}_k - \tfrac{1}{2}\hat{Y}_{Ijk}\hat{\phi}^I\hat{\psi}^{\dagger\, j}\hat{\psi}^{\dagger\, k}\,,$$

where $\hat{Y}_{Ijk} = (\hat{Y}^{Ijk})^*$ and $\hat{\phi}^I = (\hat{\phi}_I)^*$. The flavor index $I$ runs over a collection of real scalar fields $\hat{\varphi}_i$ and pairs of complex scalar fields $\hat{\Phi}_j$ and $\hat{\Phi}^j \equiv (\hat{\Phi}_j)^*$ [where a complex field and its conjugate are counted separately]. The Yukawa couplings $\hat{Y}^{Ijk}$ are symmetric under interchange of $j$ and $k$.

The mass-eigenstate basis $\psi$ is related to the interaction-eigenstate basis $\hat{\psi}$ by a unitary rotation:

$$\hat{\psi} \equiv \begin{pmatrix} \hat{\xi} \\ \hat{\chi} \\ \hat{\eta} \end{pmatrix} = U\psi \equiv \begin{pmatrix} \Omega & 0 & 0 \\ 0 & L & 0 \\ 0 & 0 & R \end{pmatrix} \begin{pmatrix} \xi \\ \chi \\ \eta \end{pmatrix}\,,$$

where $\Omega$, $L$, and $R$ are constructed as described previously. Likewise for the scalars: $\hat{\phi} = V\phi$. Thus, in terms of mass-eigenstate fields:

$$\mathscr{L}_{\rm int} = -\tfrac{1}{2}Y^{Ijk}\phi_I\psi_j\psi_k - \tfrac{1}{2}Y_{Ijk}\phi^I\psi^{\dagger\, j}\psi^{\dagger\, k}\,,$$

where $Y^{Ijk} = V_J{}^I U_m{}^j U_n{}^k \hat{Y}^{Jmn}$.

# Fermion–gauge boson interactions

In the gauge-interaction basis for the left-handed two-component fermions the corresponding interaction Lagrangian is given by

$$\mathscr{L}_{\rm int} = -g_a A^\mu_a \hat{\psi}^{\dagger\, i}\, \overline{\sigma}_\mu (\boldsymbol{T}^a)_i{}^j \hat{\psi}_j \, ,$$

where the index $a$ labels the (real or complex) vector bosons $A^\mu_a$ and is summed over. If the gauge symmetry is unbroken, then the index $a$ runs over the adjoint representation of the gauge group, and the $(\boldsymbol{T}^a)_i{}^j$ are hermitian representation matrices[†] of the gauge group acting on the left-handed fermions. There is a separate coupling $g_a$ for each simple group or U(1) factor of the gauge group $G$.

In the case of spontaneously broken gauge theories, one must diagonalize the vector boson squared mass matrix. The above form still applies where $A^a_\mu$ are gauge boson fields of definite mass, although in this case for a fixed value of $a$, $g_a\boldsymbol{T}^a$ is some linear combination of the original $g_a\boldsymbol{T}^a$ of the unbroken theory. Henceforth, we assume that that the $A^a_\mu$ are the gauge boson mass-eigenstate fields.

[†]For a $U(1)$ gauge group, the $\boldsymbol{T}^a$ are replaced by real numbers corresponding to the U(1) charges of the left-handed $(\frac{1}{2}, 0)$ fermions.

In terms of mass-eigenstate fermion fields,

$$\mathscr{L}_{\rm int} = -A^\mu_a \psi^{\dagger\, i}\, \overline{\sigma}_\mu (G^a)_i{}^j \psi_j \,,$$

where $G^a = g_a U^\dagger \boldsymbol{T}^a U$ (no sum over $a$).

Consider separately the case of gauge interactions of charged Dirac fermions. Consider pairs of left-handed $(\frac{1}{2}, 0)$ interaction-eigenstate fermions $\hat{\chi}_i$ and $\hat{\eta}^i$ that transform as conjugate representations of the gauge group (hence the difference in the flavor index heights). The Lagrangian for the gauge interactions of Dirac fermions can be written in the form:

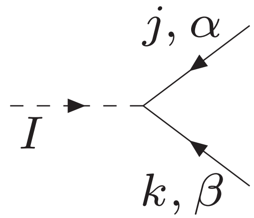
$$\mathscr{L}_{\rm int} = -g_a A^\mu_a \hat{\chi}^{\dagger\, i}\, \overline{\sigma}_\mu (\boldsymbol{T}^a)_i{}^j \hat{\chi}_j + g_a A^\mu_a \hat{\eta}^\dagger_i\, \overline{\sigma}_\mu (\boldsymbol{T}^a)_j{}^i \hat{\eta}^j \,,$$

where the $A^a_\mu$ are gauge boson mass-eigenstate fields. Here we have used the fact that if $(\boldsymbol{T}^a)_i{}^j$ are the representation matrices for the $\hat{\chi}_i$, then the $\hat{\eta}_i$ transform in the complex conjugate representation with generator matrices $-(\boldsymbol{T}^a)^* = -(\boldsymbol{T}^a)^T$. In terms of mass-eigenstate fermion fields,

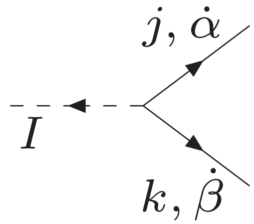
$$\mathscr{L}_{\rm int} = -A^\mu_a \left[ \chi^{\dagger\, i}\, \overline{\sigma}_\mu (G^a_L)_i{}^j \chi_j - \eta^\dagger_i\, \overline{\sigma}_\mu (G^a_R)_j{}^i \eta^j \right] \,,$$

where $G^a_L = g_a L^\dagger \boldsymbol{T}^a L$ and $G^a_R = g_a R^\dagger \boldsymbol{T}^a R$ (no sum over $a$).

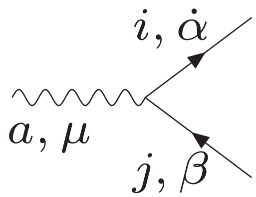
# Feynman rules for fermion interactions



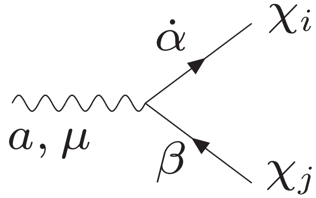

$$-iY^{Ijk}\delta_\alpha{}^\beta \quad \text{or} \quad -iY^{Ijk}\delta_\beta{}^\alpha$$



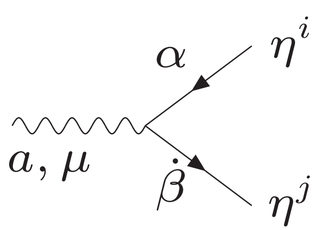

$$-iY_{Ijk}\delta^{\dot\alpha}{}_{\dot\beta} \quad \text{or} \quad -iY_{Ijk}\delta^{\dot\beta}{}_{\dot\alpha}$$




$$-i(G^a)_i{}^j\,\overline{\sigma}_\mu^{\dot\alpha\beta} \quad \text{or} \quad i(G^a)_i{}^j\,\sigma_{\mu\beta\dot\alpha}$$




$$-i(G^a_L)_i{}^j\,\overline{\sigma}_\mu^{\dot\alpha\beta} \quad \text{or} \quad ig(G^a_L)_i{}^j\,\sigma_{\mu\beta\dot\alpha}$$




$$i(G^a_R)_i{}^j\,\overline{\sigma}_\mu^{\dot\beta\alpha} \quad \text{or} \quad -ig(G^a_R)_i{}^j\,\sigma_{\mu\alpha\dot\beta}$$

# Rules for invariant amplitudes

- When computing an amplitude for a given process, all possible diagrams should be drawn that conform with the rules for external wave functions, propagators, and interactions.
- Starting from any external wave function spinor, or from any vertex on a fermion loop, factors corresponding to each propagator and vertex should be written down from left to right, following the line until it ends at another external state wave function or at the original point on the fermion loop.
- If one starts a fermion line at an $x$ or $y$ external state spinor, it should have a raised undotted index in accord with our summation conventions. Or, if one starts with an $x^\dagger$ or $y^\dagger$, it should have a lowered dotted spinor index. If one ends with an $x$ or $y$ external state spinor, it will have a lowered undotted index, while if one ends with an $x^\dagger$ or $y^\dagger$ spinor, it will have a raised dotted index. The preceding determines whether a $\sigma$ or $\overline{\sigma}$ rule should be used.
- A relative minus sign is imposed between terms contributing to a given amplitude whenever the ordering of external state spinors (written left-to-right) differs by an odd permutation.
- Each closed fermion loop gets a factor of $-1$.

With a little practice, one can write down amplitudes immediately with all spinor indices suppressed. Amplitudes generated according to these rules will contain objects of the form:

$$\mathcal{M} = z_1 \Sigma z_2$$

where $z_1$ and $z_2$ are each commuting external spinor wave functions $x$, $x^\dagger$, $y$, or $y^\dagger$, and $\Sigma$ is a sequence of alternating $\sigma$ and $\overline{\sigma}$ matrices. The complex conjugate of this quantity is given by

$$\mathcal{M}^* = z_2^\dagger \Sigma_r z_1^\dagger$$

where $\Sigma_r$ is obtained from $\Sigma$ by reversing the order of all the $\sigma$ and $\overline{\sigma}$ matrices, and using the same summation convention for suppressed spinor indices. We reiterate that:

$$^{\alpha}{}_{\alpha} \qquad \text{and} \qquad {}_{\dot\alpha}{}^{\dot\alpha}$$

governs the summation of undotted and dotted indices.

Note that different graphs contributing to the same process will often have different external state wave function spinors, with different arrow directions, for the same external fermion. Furthermore, there are no arbitrary choices to be made for arrow directions. Instead, one must add together all Feynman graphs that obey the rules.

# Self-energy functions and pole masses

After diagonalization of the fermion mass matrix, the mass-eigenstates consist of neutral Majorana fermions $\xi_k$ and Dirac fermion pairs $\chi_\ell$ and $\eta_\ell$. The tree-level fermion mass matrix, $\boldsymbol{m}^{ij}$, is made up of diagonal elements $m_k$ and $2\times 2$ blocks $\left(\begin{smallmatrix} 0 & m_\ell \\ m_\ell & 0 \end{smallmatrix}\right)$ along the diagonal, where the $m_k$ and $m_\ell$ are real and non-negative. To maintain our index summation conventions, we define $\overline{\boldsymbol{m}}_{ij} \equiv \boldsymbol{m}^{ij}$. Note that $\overline{\boldsymbol{m}}_{ik}\boldsymbol{m}^{kj} = \boldsymbol{m}^{ik}\overline{\boldsymbol{m}}_{kj} = m_i^2\delta_i^j$ is a diagonal matrix. The full, loop-corrected Feynman propagators are defined by

$$\langle 0|\, T\psi_{\alpha i}(x)\psi^{\dagger\, j}_{\dot\beta}(y)\,|0\rangle_{\rm FT} = \,ip\!\cdot\!\sigma_{\alpha\dot\beta}\,\boldsymbol{C}_i{}^j(p^2)\,,$$

$$\langle 0|\, T\psi^{\dagger\,\dot\alpha i}(x)\psi^{\beta}_{j}(y)\,|0\rangle_{\rm FT} = \,ip\!\cdot\!\overline{\sigma}^{\dot\alpha\beta}\,(\boldsymbol{C}^T)^i{}_j(p^2)\,,$$

$$\langle 0|\, T\psi^{\dagger\,\dot\alpha i}(x)\psi^{\dagger\, j}_{\dot\beta}(y)\,|0\rangle_{\rm FT} = \delta^{\dot\alpha}{}_{\dot\beta}\,\boldsymbol{D}^{ij}(p^2)\,,$$

$$\langle 0|\, T\psi_{\alpha i}(x)\psi^{\beta}_{j}(y)\,|0\rangle_{\rm FT} = \delta_\alpha{}^\beta\,\overline{\boldsymbol{D}}_{ij}(p^2)\,.$$

which can be represented diagrammatically by:


$ip\!\cdot\!\sigma_{\alpha\dot\beta}\,\boldsymbol{C}_i{}^j$ $\qquad$ $ip\!\cdot\!\overline{\sigma}^{\dot\alpha\beta}\,(\boldsymbol{C}^T)^i{}_j$ $\qquad$ $i\delta^{\dot\alpha}{}_{\dot\beta}\,\boldsymbol{D}^{ij}$ $\qquad$ $\delta_\alpha{}^\beta\,\overline{\boldsymbol{D}}_{ij}$

Starting at tree-level the full propagator functions are given by:

$$\boldsymbol{C}_i{}^j = \delta_i{}^j/(p^2 - m_i^2) + \ldots$$

$$\boldsymbol{D}^{ij} = \boldsymbol{m}^{ij}/(p^2 - m_i^2) + \ldots$$

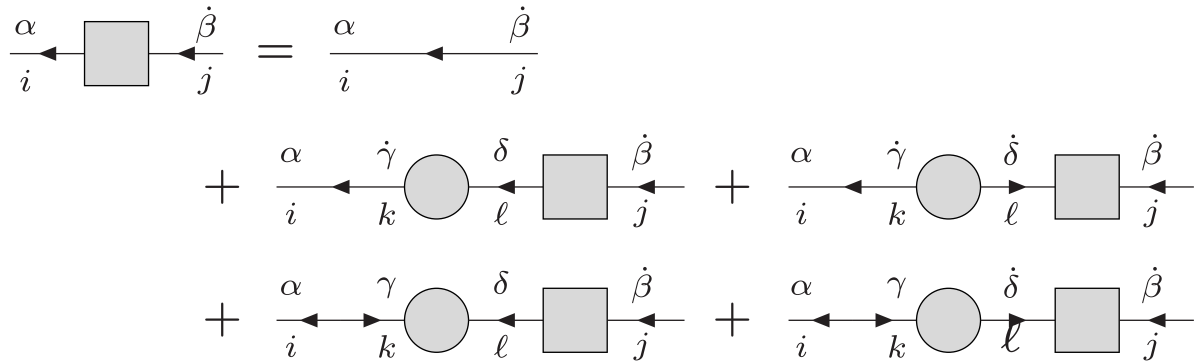
$$\overline{\boldsymbol{D}}_{ij} = \overline{\boldsymbol{m}}_{ij}/(p^2 - m_i^2) + \ldots\ ,$$

with no sum on $i$ in each case. They are functions of the external momentum invariant $p^2$ and of the masses and couplings of the theory. In general, $\boldsymbol{D}^{ij}$ is a complex symmetric matrix, and $\overline{\boldsymbol{D}}_{ij}$ can be obtained from it by taking the complex conjugate of all Lagrangian parameters appearing in its calculation, but not taking the complex conjugates of Euclideanized loop integral functions, whose imaginary (absorptive) parts correspond to fermion decay widths to multi-particle intermediate states.

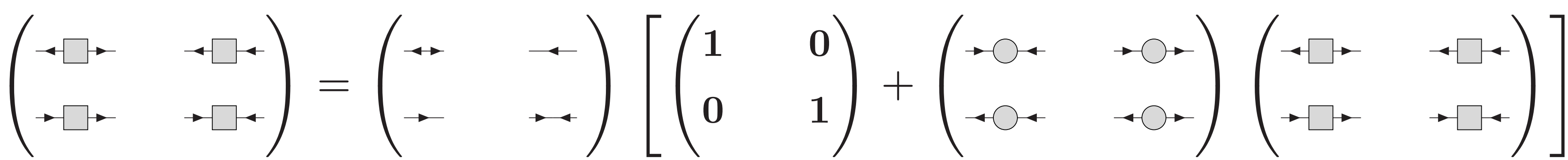
We organize the computation of the full propagators in terms of one-particle irreducible (1PI) self-energy functions, which are given by the sum of all Feynman diagrams (excluding the tree-level) that contribute to the 1PI two-point Green functions.


$-ip\cdot\overline{\sigma}^{\dot\alpha\beta}(\boldsymbol{\Xi})_i{}^j$ $\quad$ $-ip\cdot\sigma_{\alpha\dot\beta}(\boldsymbol{\Xi}^{\boldsymbol{T}})^i{}_j$ $\quad$ $-i\delta_\alpha{}^\beta\boldsymbol{\Omega}^{ij}$ $\quad$ $-i\delta^{\dot\alpha}{}_{\dot\beta}\overline{\boldsymbol{\Omega}}_{ij}$

A first diagrammatic identity (with momentum $p$ flowing from right to left):

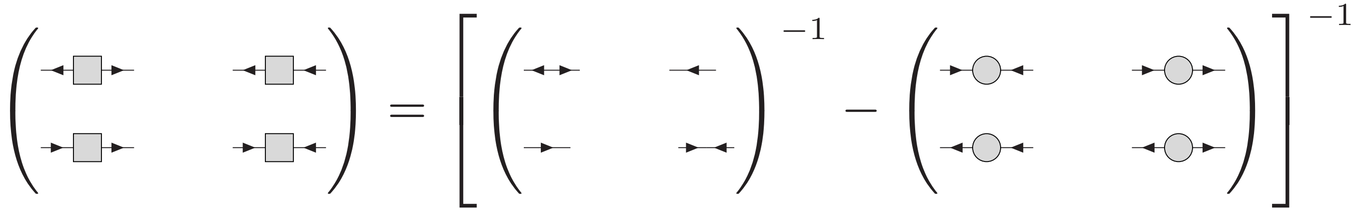


Similar diagrammatic identities can be constructed for the three other full loop-corrected propagators. The resulting four equations can be neatly summarized by the following matrix diagrammatic identity:

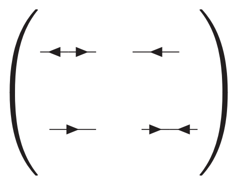


The corresponding algebraic representation can be written as $F = T + TSF$, where $F$ is the matrix of full loop-corrected propagators, $T$ is the matrix of tree-level propagators and $S$ is the matrix of self-energy functions. Multiplying on the left by $T^{-1}$ and on the right by $F^{-1}$ yields $T^{-1} = F^{-1} + S$. Thus, $F = [T^{-1} - S]^{-1}$.

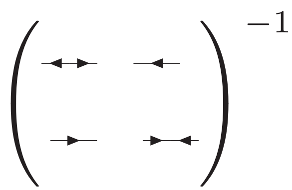
In pictures:



Explicitly, $T$ and $T^{-1}$ are given by:



$$= \frac{1}{s - m_i^2} \begin{pmatrix} i\, \overline{\boldsymbol{m}}_{ij}\, \delta_\alpha{}^\beta & ip\cdot\boldsymbol{\sigma}_{\alpha\dot\beta}\, \delta_i{}^j \\ ip\cdot\overline{\boldsymbol{\sigma}}^{\dot\alpha\beta}\, \delta^i{}_j & i\, \boldsymbol{m}^{ij}\, \delta^{\dot\alpha}{}_{\dot\beta} \end{pmatrix},$$



$$= \begin{pmatrix} i\, \boldsymbol{m}^{ij}\, \delta_\alpha{}^\beta & -ip\cdot\boldsymbol{\sigma}_{\alpha\dot\beta}\, \delta^i{}_j \\ -ip\cdot\overline{\boldsymbol{\sigma}}^{\dot\alpha\beta}\, \delta_i{}^j & i\, \overline{\boldsymbol{m}}_{ij}\, \delta^{\dot\alpha}{}_{\dot\beta} \end{pmatrix}.$$

Thus, one obtains a matrix equation for the full propagator functions:

$$\begin{pmatrix} i\overline{\boldsymbol{D}} & ip\cdot\sigma\, \boldsymbol{C} \\ ip\cdot\overline{\sigma}\, \boldsymbol{C}^T & i\boldsymbol{D} \end{pmatrix} = \begin{pmatrix} i(\boldsymbol{m} + \boldsymbol{\Omega}) & -ip\cdot\sigma\, (\boldsymbol{1} - \boldsymbol{\Xi}^T) \\ -ip\cdot\overline{\sigma}\, (\boldsymbol{1} - \boldsymbol{\Xi}) & i(\overline{\boldsymbol{m}} + \overline{\boldsymbol{\Omega}}) \end{pmatrix}^{-1},$$

where $\boldsymbol{1}$ is the $N \times N$ identity matrix.

# The pole mass

To determine the pole mass, go to rest frame of the fermion. The spinor-index dependence is now trivial. Setting $p^\mu = (\sqrt{s}\,;\,\mathbf{0})$, we search for values of $s$ where the inverse of the full propagator has a zero eigenvalue. This is equivalent to setting the determinant of the inverse of the full propagator to zero. Using the well-known formula for the determinant of a block-partitioned matrix:

$$\det \begin{pmatrix} P & Q \\ R & S \end{pmatrix} = \det P \, \det\,(S - RP^{-1}Q)\,,$$

one finds that the (in general complex) poles of the full propagator, $s_{\rm pole,j} = M_j^2 - i\Gamma_j M_j$, are the solutions to the non-linear equation:

$$\det\,\left[s\mathbf{1} - (\mathbf{1}-\boldsymbol{\Xi}^T)^{-1}(\boldsymbol{m}+\boldsymbol{\Omega})(\mathbf{1}-\boldsymbol{\Xi})^{-1}(\overline{\boldsymbol{m}}+\overline{\boldsymbol{\Omega}})\right] = 0\,.$$

with $s \equiv p^2$. This can be solved iteratively by first expanding the self-energy function matrices $\boldsymbol{\Xi}$, $\boldsymbol{\Omega}$ and $\overline{\boldsymbol{\Omega}}$ in Taylor series in $p^2$ about either $m_j^2$ or $M_j^2$. The complex pole mass quantities $s_{\rm pole,j}$ are renormalization-group and gauge invariant physical observables.

# Conventions for fermion names and fields

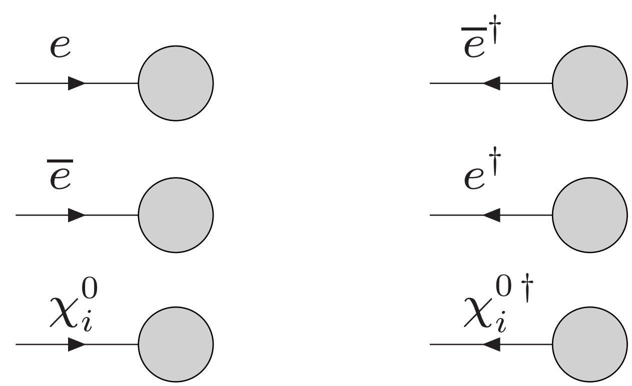
There is a one-to-one correspondence between the Majorana fermion particle names and the left-handed $(\frac{1}{2}, 0)$ fields, but for Dirac fermions there are always two distinct two-component fields that correspond to each particle name. We shall always label fermion lines with the two-component fields, rather than the particle names, with the following conventions:

- Initial-state external fermion lines (which always have physical four-momenta going into the vertex) in Feynman diagrams are labeled by the corresponding unbarred (left-handed) field if the arrow is into the vertex, and by the barred (right-handed) field if the arrow is away from the vertex.

Initial-state $e^-$:

Initial-state $e^+$:

Initial-state $\widetilde{N}_i$:

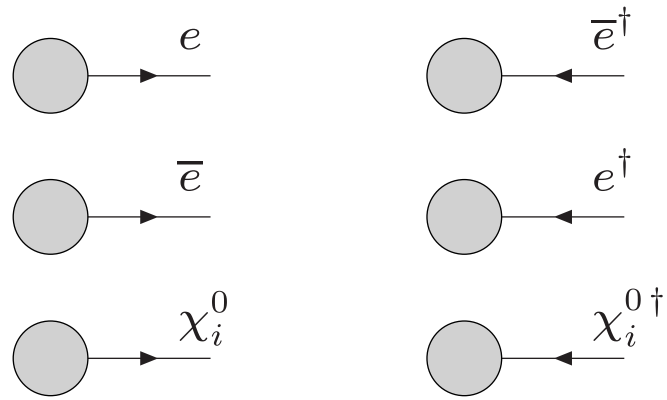


- Final-state external fermion lines in complete Feynman diagrams (which always have physical four-momenta going out of the vertex) are labeled by the corresponding barred (right-handed) field if the arrow is into the vertex, and by the unbarred (left-handed) field if the arrow is away from the vertex.

Final-state $e^-$:

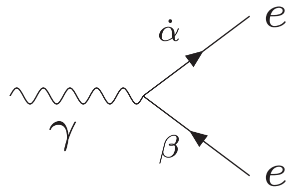
Final-state $e^+$:

Final-state $\widetilde{N}_i$:

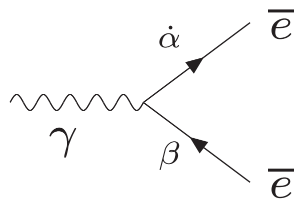



- Internal fermion lines in Feynman diagrams are also always labeled by the unbarred, left-handed field(s). Internal lines containing a propagator with opposing arrows can carry two labels.

- In the Feynman rules for interaction vertices, the external lines are always labeled by the unbarred left-handed field, regardless of its arrow direction. Two-component fermion lines with arrows going away from the vertex correspond to dotted indices, and two-component fermion lines with arrows going toward the vertex always correspond to undotted indices. This applies also to complete Feynman diagrams (e.g., self-energies) where the initial state and the final state roles are ambiguous.




$$ie\overline{\sigma}_\mu^{\dot{\alpha}\beta} \quad \text{or} \quad -ie\sigma_{\mu\beta\dot{\alpha}}$$




$$-ie\overline{\sigma}_\mu^{\dot{\alpha}\beta} \quad \text{or} \quad ie\sigma_{\mu\beta\dot{\alpha}}$$

**The two-component Feynman rules for the QED vertex**

| Fermion name | Two-component fields |
|---|---|
| $\ell^-$ (lepton) | $\ell\ ,\ \overline{\ell}^{\,\dagger}$ |
| $\ell^+$ (anti-lepton) | $\overline{\ell}\ ,\ \ell^\dagger$ |
| $\nu$ (neutrino) | $\nu\ ,\ \overline{\nu}^{\,\dagger}$ |
| $\bar{\nu}$ (antineutrino) | $\overline{\nu}\ ,\ \nu^\dagger$ |
| $q$ (quark) | $q\ ,\ \overline{q}^{\,\dagger}$ |
| $\bar{q}$ (anti-quark) | $\overline{q}\ ,\ q^\dagger$ |
| $f$ (quark or lepton) | $f\ ,\ \overline{f}^{\,\dagger}$ |
| $\bar{f}$ (anti-quark or anti-lepton) | $\overline{f}\ ,\ f^\dagger$ |
| $\widetilde{N}_i$ (neutralino) | $\chi_i^0\ ,\ \chi_i^{0\,\dagger}$ |
| $\widetilde{C}_i^+$ (chargino) | $\chi_i^+\ ,\ \chi_i^{-\,\dagger}$ |
| $\widetilde{C}_i^-$ (anti-chargino) | $\chi_i^-\ ,\ \chi_i^{+\,\dagger}$ |
| $\widetilde{g}$ (gluino) | $\widetilde{g}\ ,\ \widetilde{g}^\dagger$ |

Fermion and anti-fermion names and two-component fields in the Standard Model and the MSSM. For massive neutrinos, add $\overline{\nu}$ and $\overline{\nu}^{\,\dagger}$).

# Examples from the MSSM

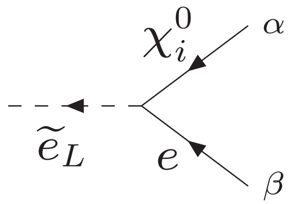
We focus on some processes involving the Majorana neutralino states. The mass matrix for these states in the $\widetilde{B}$–$\widetilde{W}^0$–$\widetilde{H}_d$–$\widetilde{H}_u$ basis is given by:

$$M_{\chi^0} = \begin{pmatrix} M_1 & 0 & -\frac{1}{2}g'v_d & \frac{1}{2}g'v_u \\ 0 & M_2 & \frac{1}{2}gv_d & -\frac{1}{2}gv_u \\ -\frac{1}{2}g'v_d & \frac{1}{2}gv_d & 0 & -\mu \\ \frac{1}{2}g'v_u & -\frac{1}{2}gv_u & -\mu & 0 \end{pmatrix},$$

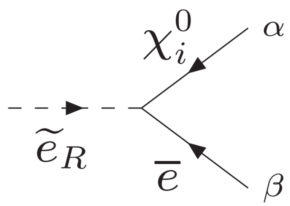
where $g' = g\tan\theta_W$, $v_u^2 + v_d^2 = (246\ \text{GeV})^2$ and $\tan\beta \equiv v_u/v_d$. The Takagi diagonalization (with unitary diagonalization matrix $N$) yields:

$$[N^{-1}]^T M_{\chi^0} N^{-1} = \text{diag}(m_{\widetilde{N}_1}, m_{\widetilde{N}_2}, m_{\widetilde{N}_3}, m_{\widetilde{N}_4}).$$

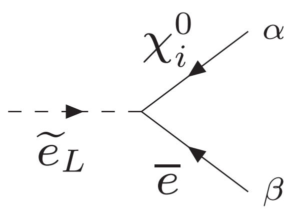
The Feynman rules for neutralino interactions with electrons and selectrons are given by:



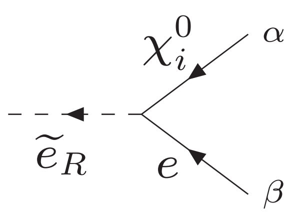

$$\frac{i}{\sqrt{2}}\left[gN_{i2}^* + g'N_{i1}^*\right]\delta_\alpha{}^\beta$$




$$-ig'\sqrt{2}\,N_{i1}^*\,\delta_\alpha{}^\beta$$




$$-i\sqrt{2}(m_e/v_d)N_{i3}^*\,\delta_\alpha{}^\beta$$

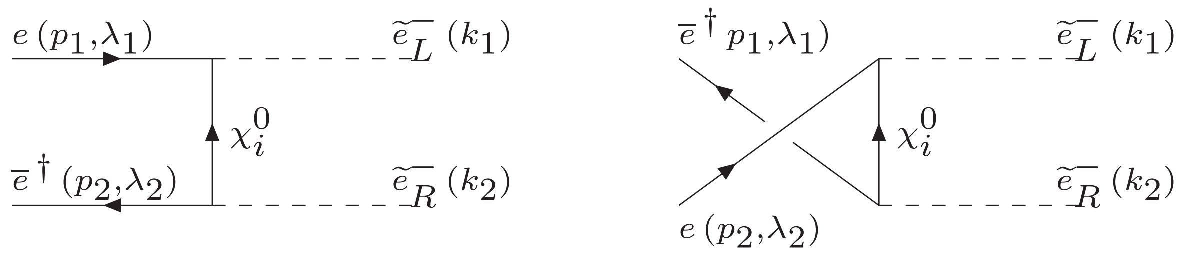



$$-i\sqrt{2}(m_e/v_d)N_{i3}^*\,\delta_\alpha{}^\beta$$

Feynman rules for the interactions of neutralinos with electron/selectron pairs in the MSSM. For each rule, there is a corresponding one with all arrows reversed, undotted indices changed to dotted indices with the opposite height, and the coupling (without the explicit $i$) replaced by its complex conjugate.

## Example 1: $e^- e^- \longrightarrow \widetilde{e}_L^- \widetilde{e}_R^-$

There are two Feynman graphs (neglecting the electron mass):



The matrix element for the first graph, for each neutralino $\widetilde{N}_i$ exchanged in the $t$ channel, is:

$$i\mathcal{M}_t = \left[ i\frac{g}{\sqrt{2}} \left( N_{i2}^* + N_{i1}^* \tan\theta_W \right) \right] \left[ -ig\sqrt{2} N_{i1} \tan\theta_W \right] x_1 \left[ \frac{i(k_1 - p_1)\cdot\sigma}{(k_1 - p_1)^2 - m_{\tilde{N}_i}^2} \right] y_2^\dagger .$$

The external wave functions are denoted by $x_i = (\vec{\boldsymbol{p}}_i, \lambda_i)$, $i = 1, 2$ and analogously for $y_i, x_i^\dagger, y_i^\dagger$. The matrix element for the second ($u$-channel) graph is related to the $t$-channel graph by the interchange of the two incoming electrons, $e_1 \leftrightarrow e_2$:

$$i\mathcal{M}_u = (-1) \left[ i\frac{g}{\sqrt{2}} \left( N_{i2}^* + N_{i1}^* \tan\theta_W \right) \right] \left[ -ig\sqrt{2} N_{i1} \tan\theta_W \right] x_2 \left[ \frac{i(k_1 - p_2)\cdot\sigma}{(k_1 - p_2)^2 - m_{\tilde{N}_i}^2} \right] y_1^\dagger .$$

Note that since we have written the fermion wave function spinors in the opposite order in $\mathcal{M}_u$ compared to $\mathcal{M}_t$, there is a factor $(-1)$ for Fermi-Dirac statistics. Alternatively, starting at the electron with momentum $p_1$ and using the $\overline{\sigma}$ rule for the propagator,

$$i\mathcal{M}_u = \left[i\frac{g}{\sqrt{2}}\left(N^*_{i2} + N^*_{i1}\tan\theta_W\right)\right]\left[-ig\sqrt{2}N_{i1}\tan\theta_W\right] y^\dagger_1 \left[\frac{-i(k_1-p_2)\cdot\overline{\sigma}}{(k_1-p_2)^2 - m^2_{\tilde{N}_i}}\right] x_2 \,.$$

Using $y^\dagger_1\overline{\sigma}x_2 = x_2\sigma y^\dagger_1$ (which holds for commuting spinors), we see that the two expressions for $\mathcal{M}_u$ coincide.

Thus, the total amplitude is given by:

$$\mathcal{M} = \mathcal{M}_t + \mathcal{M}_u = x_1 a\cdot\sigma y^\dagger_2 + y^\dagger_1 b\cdot\overline{\sigma}x_2 \,,$$

where

$$a^\mu \equiv \frac{g^2 s_W}{c_W}(k_1^\mu - p_1^\mu)\sum_{i=1}^{4} N_{i1}(N^*_{i2} + N^*_{i1}\tan\theta_W)\,\frac{1}{t - m^2_{\tilde{N}_i}}\,,$$

$$b^\mu \equiv -\frac{g^2 s_W}{c_W}(k_1^\mu - p_2^\mu)\sum_{i=1}^{4} N_{i1}(N^*_{i2} + N^*_{i1}\tan\theta_W)\,\frac{1}{u - m^2_{\tilde{N}_i}}\,,$$

and $s_W \equiv \sin\theta_W$, $c_W \equiv \cos\theta_W$, $t = (p_1 - k_1)^2$ and $u = (p_1 - k_2)^2$.

Squaring the amplitude yields

$$|\mathcal{M}|^2 = (x_1 a\cdot\sigma\bar{y}_2)\left(y_2 a^*\cdot\sigma\bar{x}_1\right) + (\bar{y}_1 b\cdot\overline{\sigma}x_2)\left(\bar{x}_2 b^*\cdot\overline{\sigma}y_1\right)$$
$$+ (x_1 a\cdot\sigma\bar{y}_2)\left(\bar{x}_2 b^*\cdot\overline{\sigma}y_1\right) + (\bar{y}_1 b\cdot\overline{\sigma}x_2)\left(y_2 a^*\cdot\sigma\bar{x}_1\right) .$$

Averaging over the initial state electron spins, the $a, b^*$ and $a^*, b$ cross terms are proportional to $m_e$ and can thus be neglected in our approximation. We get:

$$\tfrac{1}{4}\sum_{\lambda_1,\lambda_2}|\mathcal{M}|^2 = \tfrac{1}{4}\,\mathrm{Tr}[a\cdot\sigma\; p_2\cdot\overline{\sigma}\; a^*\cdot\sigma\; p_1\cdot\overline{\sigma}] + \tfrac{1}{4}\,\mathrm{Tr}[b\cdot\overline{\sigma}\; p_2\cdot\sigma\; b^*\cdot\overline{\sigma}\; p_1\cdot\sigma] .$$

Evaluating the traces yields:

$$\tfrac{1}{4}\sum_{\lambda_1,\lambda_2}|\mathcal{M}|^2 = \tfrac{1}{4}g^4\tan^2\theta_W(tu - m^2_{\tilde{e}_L}m^2_{\tilde{e}_R})$$
$$\times\sum_{i,j=1}^{4} N_{j1}N^*_{i1}(N^*_{j2} + N^*_{j1}\tan\theta_W)(N_{i2} + N_{i1}\tan\theta_W)$$
$$\times\left[\frac{1}{(t - m^2_{\tilde{N}_i})(t - m^2_{\tilde{N}_j})} + \frac{1}{(u - m^2_{\tilde{N}_i})(u - m^2_{\tilde{N}_j})}\right] .$$

In obtaining these results, we have employed the identities:

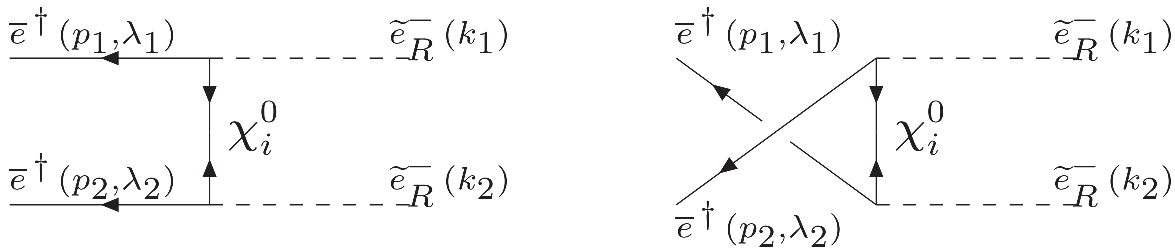
$$\mathrm{Tr}[(k_1 - p_1)\cdot\sigma\ p_2\cdot\overline{\sigma}\ (k_1 - p_1)\cdot\sigma\ p_1\cdot\overline{\sigma}] = \mathrm{Tr}[(k_1 - p_2)\cdot\overline{\sigma}\ p_2\cdot\sigma\ (k_1 - p_2)\cdot\overline{\sigma}\ p_1\cdot\sigma]$$
$$= tu - m^2_{\tilde{e}_L} m^2_{\tilde{e}_R},$$

Thus, for $s = (p_1 + p_2)^2$, the differential cross-section is:

$$\frac{d\sigma}{dt} = \frac{\pi\alpha^2}{4s_W^2 c_W^2}\left(\frac{tu - m^2_{\tilde{e}_L} m^2_{\tilde{e}_R}}{s^2}\right)$$
$$\times \sum_{i,j=1}^{4} N_{j1} N^*_{i1} (N^*_{j2} + N^*_{j1}\tan\theta_W)(N_{i2} + N_{i1}\tan\theta_W)$$
$$\times \left[\frac{1}{(t - m^2_{\tilde{N}_i})(t - m^2_{\tilde{N}_j})} + \frac{1}{(u - m^2_{\tilde{N}_i})(u - m^2_{\tilde{N}_j})}\right] .$$

## Example 2: $e^-e^- \to \widetilde{e}_R^-\widetilde{e}_R^-$

Again, in the limit of vanishing electron mass, there are two Feynman graphs, which are related by the exchange of identical electrons in the initial state or equivalently by exchange of the identical selectrons in the final state.




The amplitude for the first graph is:

$$i\mathcal{M}_t = \left(-ig\sqrt{2}N_{i1}\tan\theta_W\right)^2\left[\frac{i\,m_{\widetilde{N}_i}}{(k_1-p_1)^2-m^2_{\widetilde{N}_i}}\right]y_1^\dagger y_2^\dagger\,,$$

for each exchanged neutralino. The amplitude for the second graph is:

$$i\mathcal{M}_u = \left(-ig\sqrt{2}N_{i1}\tan\theta_W\right)^2\left[\frac{i\,m_{\widetilde{N}_i}}{(k_1-p_2)^2-m^2_{\widetilde{N}_i}}\right]y_1^\dagger y_2^\dagger\,.$$

Since we have chosen to write the external state wave function spinors in the same order in $\mathcal{M}_t$ and $\mathcal{M}_u$, there is no factor of $(-1)$ for Fermi-Dirac statistics.

The total amplitude squared is:

$$|\mathcal{M}|^2 = 4g^4 \tan^4 \theta_W (y_1^\dagger y_2^\dagger)(y_2 y_1) \sum_{i,j=1}^{4} (N_{i1})^2 (N_{j1}^*)^2 m_{\widetilde{N}_i} m_{\widetilde{N}_j}$$

$$\times \left( \frac{1}{t - m_{\widetilde{N}_i}^2} + \frac{1}{u - m_{\widetilde{N}_i}^2} \right) \left( \frac{1}{t - m_{\widetilde{N}_j}^2} + \frac{1}{u - m_{\widetilde{N}_j}^2} \right)$$

The sum over the electron spins is obtained from

$$\sum_{\lambda_1, \lambda_2} (\bar{y}_1 \bar{y}_2)(y_2 y_1) = \mathrm{Tr}[p_2 \cdot \overline{\sigma} p_1 \cdot \sigma] = 2 p_2 \cdot p_1 = s \ .$$

Hence, the spin-averaged differential cross-section is:

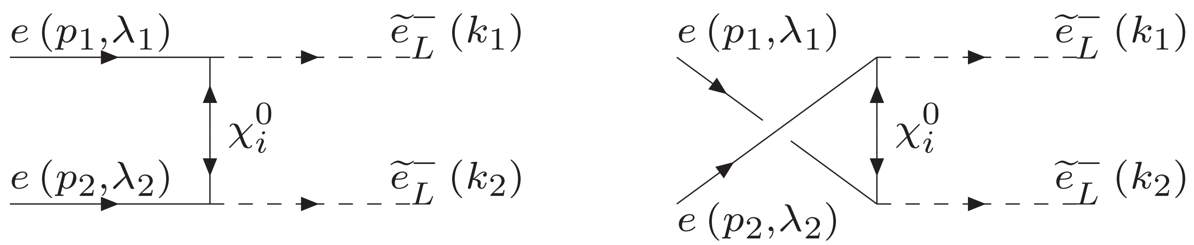
$$\frac{d\sigma}{dt} = \frac{\pi \alpha^2}{c_W^4} \sum_{i,j=1}^{4} (N_{i1})^2 (N_{j1}^*)^2 \frac{m_{\widetilde{N}_i} m_{\widetilde{N}_j}}{s}$$

$$\times \left( \frac{1}{t - m_{\widetilde{N}_i}^2} + \frac{1}{u - m_{\widetilde{N}_i}^2} \right) \left( \frac{1}{t - m_{\widetilde{N}_j}^2} + \frac{1}{u - m_{\widetilde{N}_j}^2} \right)$$

Note that when integrating over the $4\pi$ solid angle to obtain the total cross-section, one must multiply by a factor of $1/2$ due to the identical sleptons in the final state.

# Example 3: $e^- e^- \to \widetilde{e}_L^- \widetilde{e}_L^-$

Again, in the limit of vanishing electron mass, there are two Feynman graphs, which are related by the exchange of identical electrons in the initial state or equivalently by exchange of the identical selectrons in the final state. The contributing graphs are exactly like the previous example, but with all arrows reversed.




The computation of the invariant amplitude and cross-section is very similar to the previous example, so the details will be omitted here.

# Four-component spinor notation

The correspondence between the two-component and four-component spinor language is most easily exhibited in the basis in which $\gamma_5$ is diagonal (this is called the *chiral* representation). In 2×2 blocks, the gamma matrices are given by:

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu_{\alpha\dot\beta} \\ \overline{\sigma}^{\mu\dot\alpha\beta} & 0 \end{pmatrix}\,, \qquad \gamma_5 \equiv i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} -\delta_\alpha{}^\beta & 0 \\ 0 & \delta^{\dot\alpha}{}_{\dot\beta} \end{pmatrix}\,.$$

The chiral projections operators are: $P_L \equiv \frac{1}{2}(1-\gamma_5)$ and $P_R \equiv \frac{1}{2}(1+\gamma_5)$.

In addition, we identify the generators of the Lorentz group in the $(\frac{1}{2},0)\oplus(0,\frac{1}{2})$ representation:[‡]

$$\tfrac{1}{2}\Sigma^{\mu\nu} \equiv \frac{i}{4}[\gamma^\mu,\gamma^\nu] = \begin{pmatrix} \sigma^{\mu\nu}{}_\alpha{}^\beta & 0 \\ 0 & \overline{\sigma}^{\mu\nu\dot\alpha}{}_{\dot\beta} \end{pmatrix}\,.$$

[‡]In most textbooks, $\Sigma^{\mu\nu}$ is called $\sigma^{\mu\nu}$. Here, we use the former symbol so that there is no confusion with the two-component definition of $\sigma^{\mu\nu}$.

A four component Dirac spinor field, $\Psi(x)$, is made up of two mass-degenerate two-component spinor fields, $\chi_\alpha(x)$ and $\eta_\alpha(x)$ as follows:

$$\Psi(x) \equiv \begin{pmatrix} \chi_\alpha(x) \\ \\ \eta^{\dagger\,\dot{\alpha}}(x) \end{pmatrix} .$$

Note that $P_L$ and $P_R$ project out the upper and lower components, respectively. The Dirac conjugate field $\overline{\Psi}$ and the charge conjugate field $\Psi^c$ are defined by

$$\overline{\Psi}(x) \equiv \Psi^\dagger A = (\eta^\alpha(x), \chi^\dagger_{\dot{\alpha}}) ,$$

$$\Psi^c(x) \equiv C\overline{\Psi}^T(x) = \begin{pmatrix} \eta_\alpha(x) \\ \\ \chi^{\dagger\,\dot{\alpha}}(x) \end{pmatrix} ,$$

where the Dirac conjugation matrix $A$ and the charge conjugation matrix $C$ satisfy

$$A\gamma^\mu A^{-1} = \gamma^{\mu\dagger} , \qquad\qquad C^{-1}\gamma^\mu C = -\gamma^{\mu T} .$$

In the chiral representation, $A$ and $C$ are explicitly given by

$$A = \begin{pmatrix} 0 & \delta^{\dot{\alpha}}{}_{\dot{\beta}} \\ \delta_{\alpha}{}^{\beta} & 0 \end{pmatrix} , \qquad C = -\gamma_5 B^{-1} = \begin{pmatrix} \epsilon_{\alpha\beta} & 0 \\ 0 & \epsilon^{\dot{\alpha}\dot{\beta}} \end{pmatrix} .$$

Note the numerical equalities, $A = \gamma^0$ and $C = i\gamma^0\gamma^2$, although these identifications do not respect the structure of the undotted and dotted indices specified above.

One can relate bilinear covariants in two-component and four-component notation.

$$\overline{\Psi}_1\Psi_2 = \eta_1\xi_2 + \xi_1^\dagger\eta_2^\dagger$$

$$\overline{\Psi}_1\gamma_5\Psi_2 = -\eta_1\xi_2 + \xi_1^\dagger\eta_2^\dagger$$

$$\overline{\Psi}_1\gamma^\mu\Psi_2 = \xi_1\overline{\sigma}^\mu\xi_2 - \eta_2^\dagger\overline{\sigma}^\mu\eta_1$$

$$\overline{\Psi}_1\gamma^\mu\gamma_5\Psi_2 = -\xi_1^\dagger\overline{\sigma}^\mu\xi_2 - \eta_2^\dagger\overline{\sigma}^\mu\eta_1$$

$$\overline{\Psi}_1\Sigma^{\mu\nu}\Psi_2 = 2(\eta_1\sigma^{\mu\nu}\xi_2 + \xi_1^\dagger\overline{\sigma}^{\mu\nu}\eta_2^\dagger)$$

$$\overline{\Psi}_1\Sigma^{\mu\nu}\gamma_5\Psi_2 = -2(\eta_1\sigma^{\mu\nu}\xi_2 - \xi_1^\dagger\overline{\sigma}^{\mu\nu}\eta_2^\dagger) .$$

## Relating bilinear covariants in two-component and four-component notation

$$\Psi_1(x) \equiv \begin{pmatrix} \xi_1(x) \\ \eta_1^\dagger(x) \end{pmatrix}, \qquad \Psi_2(x) \equiv \begin{pmatrix} \xi_2(x) \\ \eta_2^\dagger(x) \end{pmatrix}.$$

| | |
|---|---|
| $\overline{\Psi}_1 P_L \Psi_2 = \eta_1 \xi_2$ | $\overline{\Psi}{}^c_1 P_L \Psi^c_2 = \xi_1 \eta_2$ |
| $\overline{\Psi}_1 P_R \Psi_2 = \xi_1^\dagger \eta_2^\dagger$ | $\overline{\Psi}{}^c_1 P_R \Psi^c_2 = \eta_1^\dagger \xi_2^\dagger$ |
| $\overline{\Psi}{}^c_1 P_L \Psi_2 = \xi_1 \xi_2$ | $\overline{\Psi}_1 P_L \Psi^c_2 = \eta_1 \eta_2$ |
| $\overline{\Psi}_1 P_R \Psi^c_2 = \xi_1^\dagger \xi_2^\dagger$ | $\overline{\Psi}{}^c_1 P_R \Psi_2 = \eta_1^\dagger \eta_2^\dagger$ |
| $\overline{\Psi}_1 \gamma^\mu P_L \Psi_2 = \xi_1^\dagger \overline{\sigma}^\mu \xi_2$ | $\overline{\Psi}{}^c_1 \gamma^\mu P_L \Psi^c_2 = \eta_1^\dagger \overline{\sigma}^\mu \eta_2$ |
| $\overline{\Psi}{}^c_1 \gamma^\mu P_R \Psi^c_2 = \xi_1 \sigma^\mu \xi_2^\dagger$ | $\overline{\Psi}_1 \gamma^\mu P_R \Psi_2 = \eta_1 \sigma^\mu \eta_2^\dagger$ |
| $\overline{\Psi}_1 \Sigma^{\mu\nu} P_L \Psi_2 = 2\, \eta_1 \sigma^{\mu\nu} \xi_2$ | $\overline{\Psi}{}^c_1 \Sigma^{\mu\nu} P_L \Psi^c_2 = 2\, \xi_1 \sigma^{\mu\nu} \eta_2$ |
| $\overline{\Psi}_1 \Sigma^{\mu\nu} P_R \Psi_2 = 2\, \xi_1^\dagger \overline{\sigma}^{\mu\nu} \eta_2^\dagger$ | $\overline{\Psi}{}^c_1 \Sigma^{\mu\nu} P_R \Psi^c_2 = 2\, \eta_1^\dagger \overline{\sigma}^{\mu\nu} \xi_2^\dagger$ |

$\Sigma^{\mu\nu} \equiv \frac{i}{2}[\gamma^\mu, \gamma^\nu]$. Note that we may also write: $\overline{\Psi}_1 \gamma^\mu P_R \Psi_2 = -\eta_2 \overline{\sigma}^\mu \eta_1^\dagger$, *etc.*

For Majorana fermions defined by $\Psi_M = \Psi^c_M = C\overline{\Psi}^T_M$, the following additional conditions are satisfied:

$$\overline{\Psi}_{M1}\Psi_{M2} = \overline{\Psi}_{M2}\Psi_{M1}\,,$$

$$\overline{\Psi}_{M1}\gamma_5\Psi_{M2} = \overline{\Psi}_{M2}\gamma_5\Psi_{M1}\,,$$

$$\overline{\Psi}_{M1}\gamma^\mu\Psi_{M2} = -\overline{\Psi}_{M2}\gamma^\mu\Psi_{M1}\,,$$

$$\overline{\Psi}_{M1}\gamma^\mu\gamma_5\Psi_{M2} = \overline{\Psi}_{M2}\gamma^\mu\gamma_5\Psi_{M1}\,,$$

$$\overline{\Psi}_{M1}\Sigma^{\mu\nu}\Psi_{M2} = -\overline{\Psi}_{M2}\Sigma^{\mu\nu}\Psi_{M1}\,,$$

$$\overline{\Psi}_{M1}\Sigma^{\mu\nu}\gamma_5\Psi_{M2} = -\overline{\Psi}_{M2}\Sigma^{\mu\nu}\gamma_5\Psi_{M1}\,.$$

In particular, if $\Psi_{M1} = \Psi_{M2} \equiv \Psi_M$, then

$$\overline{\Psi}_M\gamma^\mu\Psi_M = \overline{\Psi}_M\Sigma^{\mu\nu}\Psi_M = \overline{\Psi}_M\Sigma^{\mu\nu}\gamma_5\Psi_M = 0\,.$$

One additional useful result is:

$$\overline{\Psi}_{M1}\gamma^\mu P_L\Psi_{M2} = -\overline{\Psi}_{M2}\gamma^\mu P_R\Psi_{M1}\,.$$

# Four-component spinor wave functions

The two-component spinor wave functions are related to the traditional four-component spinors according to:

$$u(\vec{\boldsymbol{p}}, s) = \begin{pmatrix} x_\alpha(\vec{\boldsymbol{p}}, s) \\ y^{\dagger\,\dot{\alpha}}(\vec{\boldsymbol{p}}, s) \end{pmatrix} , \quad \bar{u}(\vec{\boldsymbol{p}}, s) = \left(y^\alpha(\vec{\boldsymbol{p}}, s),\; x^\dagger_{\dot{\alpha}}(\vec{\boldsymbol{p}}, s)\right) ,$$

$$v(\vec{\boldsymbol{p}}, s) = \begin{pmatrix} y_\alpha(\vec{\boldsymbol{p}}, s) \\ x^{\dagger\,\dot{\alpha}}(\vec{\boldsymbol{p}}, s) \end{pmatrix} , \quad \bar{v}(\vec{\boldsymbol{p}}, s) = \left(x^\alpha(\vec{\boldsymbol{p}}, s),\; y^\dagger_{\dot{\alpha}}(\vec{\boldsymbol{p}}, s)\right) ,$$

where $v(\vec{\boldsymbol{p}}, s) = C\bar{u}(\vec{\boldsymbol{p}}, s)^T$, and $s = \pm\frac{1}{2}$. The spinor wave functions $u$ and $v$ satisfy the Dirac equation,

$$(\not{p} - m)\, u(\vec{\boldsymbol{p}}, s) = (\not{p} + m)\, v(\vec{\boldsymbol{p}}, s) = 0\,.$$

where $\not{p} \equiv \gamma_\mu p^\mu$. For massive fermions, we also have:

$$(2s\gamma_5 \not{S} - 1)\, u(\vec{\boldsymbol{p}}, s) = (2s\gamma_5 \not{S} - 1)\, v(\vec{\boldsymbol{p}}, s) = 0\,.$$

The spin projection operators for massive fermions read:

$$u(\vec{\boldsymbol{p}}, s)\bar{u}(\vec{\boldsymbol{p}}, s) = \tfrac{1}{2}(1 + 2s\gamma_5 \not{S})\,(\not{p} + m)\,,$$

$$v(\vec{\boldsymbol{p}}, s)\bar{v}(\vec{\boldsymbol{p}}, s) = \tfrac{1}{2}(1 + 2s\gamma_5 \not{S})\,(\not{p} - m)\,.$$

The corresponding Bouchiat-Michel formulae are:

$$u(\vec{\boldsymbol{p}}, s')\bar{u}(\vec{\boldsymbol{p}}, s) = \tfrac{1}{2}\left[\delta_{ss'} + \gamma_5 \not{S}_{ss'}\right](\not{p} + m)\,,$$

$$v(\vec{\boldsymbol{p}}, s')\bar{v}(\vec{\boldsymbol{p}}, s) = \tfrac{1}{2}\left[\delta_{s's} + \gamma_5 \not{S}_{s's}\right](\not{p} - m)\,,$$

For massless fermions, the helicity spinors are eigenstates of $\gamma_5$

$$\gamma_5 u(\vec{\boldsymbol{p}}, \lambda) = 2\lambda u(\vec{\boldsymbol{p}}, \lambda)\,, \quad \gamma_5 v(\vec{\boldsymbol{p}}, \lambda) = -2\lambda v(\vec{\boldsymbol{p}}, \lambda)\,.$$

The latter result can be derived from the former by putting $S^\mu = p^\mu/m + \mathcal{O}(m/E)$ and applying the Dirac equation before taking the $m \to 0$ limit. Thus, the massless fermion helicity projection operators are:

$$u(\vec{\boldsymbol{p}}, \lambda)\bar{u}(\vec{\boldsymbol{p}}, \lambda) = \tfrac{1}{2}(1 + 2\lambda\gamma_5)\,\not{p}\,,$$

$$v(\vec{\boldsymbol{p}}, \lambda)\bar{v}(\vec{\boldsymbol{p}}, \lambda) = \tfrac{1}{2}(1 - 2\lambda\gamma_5)\,\not{p}\,.$$

The corresponding Bouchiat-Michel formulae are:

$$u(p,\lambda')\bar{u}(p,\lambda) = \tfrac{1}{2}(1+2\lambda\gamma_5)\,\not{p}\,\delta_{\lambda\lambda'} + \tfrac{1}{2}\gamma_5[\not{S}^1\tau^1_{\lambda\lambda'} + \not{S}^2\tau^2_{\lambda\lambda'}]\,\not{p}\,,$$

$$v(p,\lambda')\bar{v}(p,\lambda) = \tfrac{1}{2}(1-2\lambda\gamma_5)\,\not{p}\,\delta_{\lambda'\lambda} + \tfrac{1}{2}\gamma_5[\not{S}^1\tau^1_{\lambda'\lambda} + \not{S}^2\tau^2_{\lambda'\lambda}]\,\not{p}\,.$$

Finally, the spin-sum identities are given by:

$$\sum_s u(\vec{\boldsymbol{p}},s)\bar{u}(\vec{\boldsymbol{p}},s) = \not{p} + m\,,$$

$$\sum_s v(\vec{\boldsymbol{p}},s)\bar{v}(\vec{\boldsymbol{p}},s) = \not{p} - m\,,$$

$$\sum_s u(\vec{\boldsymbol{p}},s)v^T(\vec{\boldsymbol{p}},s) = (\not{p} + m)C^T\,,$$

$$\sum_s \bar{u}^T(\vec{\boldsymbol{p}},s)\bar{v}(\vec{\boldsymbol{p}},s) = C^{-1}(\not{p} - m)\,,$$

$$\sum_s \bar{v}^T(\vec{\boldsymbol{p}},s)\bar{u}(\vec{\boldsymbol{p}},s) = C^{-1}(\not{p} + m)\,,$$

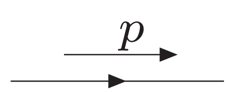
$$\sum_s v(\vec{\boldsymbol{p}},s)u^T(\vec{\boldsymbol{p}},s) = (\not{p} - m)C^T\,.$$

# Rules for four-component Majorana fermions

Consider first the Feynman rule for the four-component fermion propagator. Virtual Dirac fermion lines can either correspond to $\Psi$ or $\Psi^c$. Here, there is no ambiguity in the propagator Feynman rule, since for free Dirac fermion fields,

$$\langle 0|\, T[\Psi(x)\overline{\Psi}^{\,b}(y)]\,|0\rangle = \langle 0|\, T[\Psi^c(x)\overline{\Psi^c}(y)]\,|0\rangle \ ,$$

so that the Feynman rules for the propagator of a $\Psi$ and $\Psi^c$ line, exhibited below, are identical. The same rule also applies to a four-component Majorana fermion.



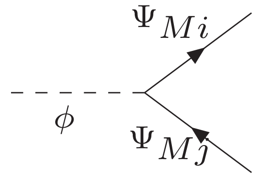

$$\frac{i(\not{p}+m)}{p^2-m^2+i\epsilon}$$

Consider a set of neutral Majorana and charged Dirac fermions interacting with a neutral scalar or vector boson. The interaction Lagrangian in terms of two-component fermions is:

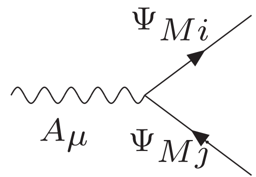
$$\mathcal{L}_{\rm int} = -\tfrac{1}{2}(\lambda^{ij}\xi_i\xi_j + \lambda_{ij}\xi^{\dagger\,i}\xi^{\dagger\,j})\phi - (\kappa^i{}_j\chi_i\eta^j + \kappa_i{}^j\chi^{\dagger\,i}\eta^\dagger_j)\phi$$

$$-G_i{}^j\,\xi^{\dagger\,i}\overline{\sigma}^\mu\xi_j A_\mu - [(G_L)_i{}^j\chi^{\dagger\,i}\overline{\sigma}^\mu\chi_j + (G_R)_i{}^j\eta^{\dagger\,i}\overline{\sigma}^\mu\eta_j]A_\mu\,,$$

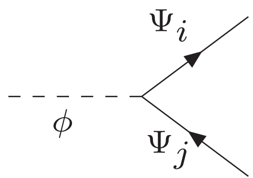
where $\lambda$ is a complex symmetric matrix with $\lambda^{ij} = \lambda^*_{ij}$, $\kappa$ is an arbitrary complex matrix with $\kappa_i{}^j = (\kappa^i{}_j)^*$, and $G$, $G_L$ and $G_R$ are hermitian matrices. Converting to four-component spinors, the Feynman rules are:



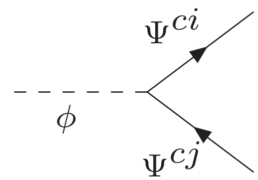

$$-i(\lambda^{ij} P_L + \lambda_{ij} P_R)$$



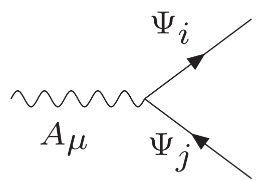

$$-i\gamma_\mu [G_i{}^j P_L - G_j{}^i P_R]$$




or

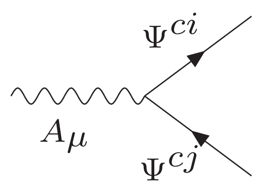



$$-i(\kappa^i{}_j P_L + \kappa_j{}^i P_R)$$




$$-i\gamma_\mu [(G_L)_i{}^j P_L + (G_R)_i{}^j P_R]$$

or or




$$i\gamma_\mu [(G_L)_i{}^j P_L + (G_R)_i{}^j P_R]$$

The arrows on the Dirac fermion lines depict the flow of the conserved charge. A Majorana fermion is self-conjugate, so its arrow simply reflects the structure of $\mathcal{L}_{\rm int}$; *i.e.*, $\overline{\Psi}_M$ [$\Psi_M$] is represented by an arrow pointing out of [into] the vertex. The arrow directions determine the placement of the $u$ and $v$ spinors in an invariant amplitude.

For vertices involving Dirac fermions, one has a choice of either using the Dirac field or its charge conjugated field. The Feynman rules corresponding to these two choices are related, due to the following identity satisfied by anti-commuting fields,

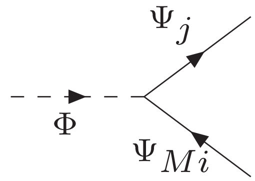
$$\overline{\Psi}{}^c_i\Gamma\Psi^c_j = -\Psi^T_i C^{-1}\Gamma C\overline{\Psi}{}^T_j = \overline{\Psi}_j C\Gamma^T C^{-1}\Psi_i = \eta_\Gamma\overline{\Psi}_j\Gamma\Psi_i\,,$$

where the sign $\eta_\Gamma = +1$ for $\Gamma = 1, \gamma_5, \gamma^\mu\gamma_5$ and $\eta_\Gamma = -1$ for $\Gamma = \gamma^\mu$, $\Sigma^{\mu\nu}$, $\Sigma^{\mu\nu}\gamma_5$.

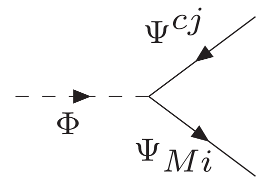
Next, consider the interaction of fermions with charged bosons, where the charges of $\Phi$, $W$ and $\chi$ are assumed to be equal. The corresponding interaction Lagrangian is given by:

$$\begin{aligned}\mathscr{L}_{\rm int} = &-\Phi[(\kappa_1)^i{}_j\xi_i\eta^j + (\kappa_2)_{ij}\xi^{\dagger i}\chi^{\dagger j}] - \Phi^\dagger[(\kappa_2)^{ij}\xi_i\chi_j + (\kappa_1)_i{}^j\xi^{\dagger i}_i\eta^\dagger_j] \\ &-W_\mu[(G_1)_j{}^i\chi^{\dagger j}\overline{\sigma}^\mu\xi_i - (G_2)_{ij}\xi^{\dagger i}\overline{\sigma}^\mu\eta^j] - W^\dagger_\mu[(G_1)^j{}_i\xi^{\dagger i}\overline{\sigma}^\mu\chi_j - (G_2)^{ij}\eta^\dagger_j\overline{\sigma}^\mu\xi_i]\,,\end{aligned}$$

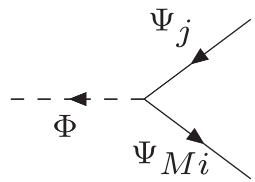
where $\kappa_1$, $\kappa_2$, $G_1$ and $G_2$ are complex matrices. Converting to four-component spinors, the corresponding Feynman rules are:




or




$$-i({\kappa_1}^{i}{}_{j} P_L + \kappa_{2ij} P_R)$$

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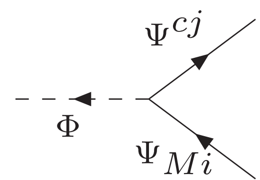
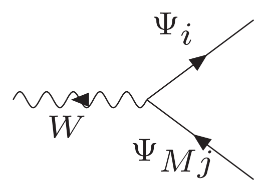
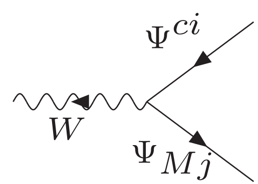



$$-i({\kappa_2}^{ij} P_L + {\kappa_{1i}}^{j} P_R)$$

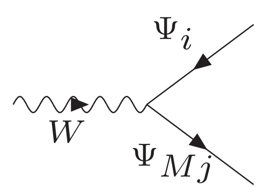



$$-i\gamma^\mu({G_{1i}}^{j} P_L - G_{2ji} P_R)$$

or

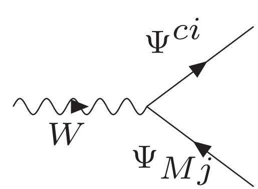



$$i\gamma^\mu(G_{2ji} P_R - {G_{1i}}^{j} P_L)$$




$$-i\gamma^\mu({G_1}^{i}{}_{j} P_L - {G_2}^{ji} P_R)$$

or




$$i\gamma^\mu({G_2}^{ji} P_R - {G_1}^{i}{}_{j} P_L)$$

## Labeling fermion lines in Feynman diagrams

One is free to choose either a $\Psi$ or $\Psi^c$ line to represent a Dirac fermion at any place in a given Feynman graph. The direction of the arrow on the $\Psi$ or $\Psi^c$ line indicates the corresponding direction of charge flow.[§] Moreover, the structure of $\mathcal{L}_{\rm int}$ implies that the arrow directions on fermion lines flow continuously through the diagram. This requirement then determines the direction of the arrows on Majorana fermion lines.

The Feynman rules for the external fermion wave functions are the same for Dirac and Majorana fermions:

- $u(\vec{\boldsymbol{p}}, s)$: incoming $\Psi$ [or $\Psi^c$] with momentum $\vec{\boldsymbol{p}}$ parallel to the arrow direction,
- $\bar{u}(\vec{\boldsymbol{p}}, s)$: outgoing $\Psi$ [or $\Psi^c$] with momentum $\vec{\boldsymbol{p}}$ parallel to the arrow direction,
- $v(\vec{\boldsymbol{p}}, s)$: outgoing $\Psi$ [or $\Psi^c$] with momentum $\vec{\boldsymbol{p}}$ anti-parallel to the arrow direction,
- $\bar{v}(\vec{\boldsymbol{p}}, s)$: incoming $\Psi$ [or $\Psi^c$] with momentum $\vec{\boldsymbol{p}}$ anti-parallel to the arrow direction.

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[§]Since the charge of $\Psi^c$ is opposite to that of $\Psi$, the corresponding arrow direction of the two lines point in opposite directions.

## Construction of invariant amplitudes involving Majorana fermions

When computing an invariant amplitude, one first writes down the relevant Feynman diagrams with no arrows on any Majorana fermion line. The number of distinct graphs contributing to the process is then determined. Finally, one makes some choice for how to distribute the arrows on the Majorana fermion lines and how to label Dirac fermion lines (either $\Psi$ or $\Psi^c$) in a manner consistent with the Feynman rules for the interaction vertices. The end result for the invariant amplitude (apart from an overall unobservable phase) does not depend on the choices made for the direction of the fermion arrows.

Example 1: $\Psi(p_1)\Psi(p_2) \longrightarrow \Phi(k_1)\Phi(k_2)$ via $\Psi_M$-exchange

The contributing Feynman graphs are:


Following the arrows in reverse, the invariant amplitude is easily computed.

$$i\mathcal{M} = (-i)^2 \bar{v}(\vec{\boldsymbol{p}}_2, s_2)(\kappa_1 P_L + \kappa_2^* P_R) \left[\frac{i(\not{p}_1 - \not{k}_1 + m)}{t - m^2} + \frac{i(\not{k}_1 - \not{p}_2 + m)}{u - m^2}\right] (\kappa_1 P_L + \kappa_2^* P_R) u(\vec{\boldsymbol{p}}_1, s_1) \,,$$

where $t \equiv (p_1 - k_1)^2$, $u \equiv (p_2 - k_1)^2$ and $m$ is the Majorana fermion mass. The sign of each diagram is determined simply by the relative permutation of spinor factors appearing in the amplitude (the overall sign of the amplitude is unphysical).

Exercise: Check that $i\mathcal{M}$ is antisymmetric under interchange of the two initial electrons. *HINT:* Taking the transpose and using $v \equiv u^c \equiv C\bar{u}^T$ (the $u$ and $v$ spinors are commuting objects), one easily verifies that:

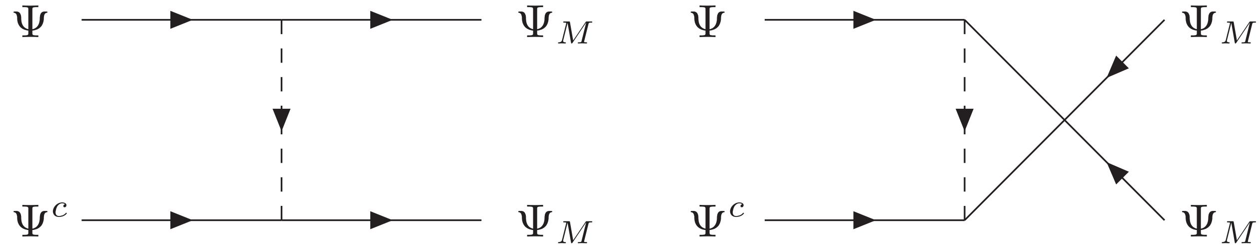
$$\bar{v}(\vec{\boldsymbol{p}}_2, s_2)\Gamma u(\vec{\boldsymbol{p}}_1, s_1) = -\eta_\Gamma \bar{v}(\vec{\boldsymbol{p}}_1, s_1)\Gamma u(\vec{\boldsymbol{p}}_2, s_2) \,,$$

where as before $\eta_\Gamma = +1$ for $\Gamma = 1, \gamma_5, \gamma^\mu\gamma_5$ and $\eta_\Gamma = -1$ for $\Gamma = \gamma^\mu, \Sigma^{\mu\nu}, \Sigma^{\mu\nu}\gamma_5$.

Example 2: $\Psi(p_1)\Psi^c(p_2) \to \Psi_M(p_3)\Psi_M(p_4)$ via charged $\Phi$-exchange

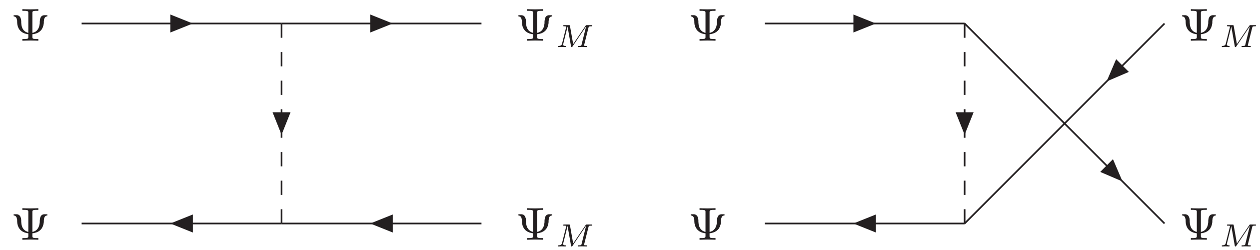
Neglecting a possible $s$-channel annihilation graph, the contributing Feynman graphs can be represented either by one of two sets of diagrams.

Diagram set (i):




or an alternative set of diagrams:

Diagram set (ii):

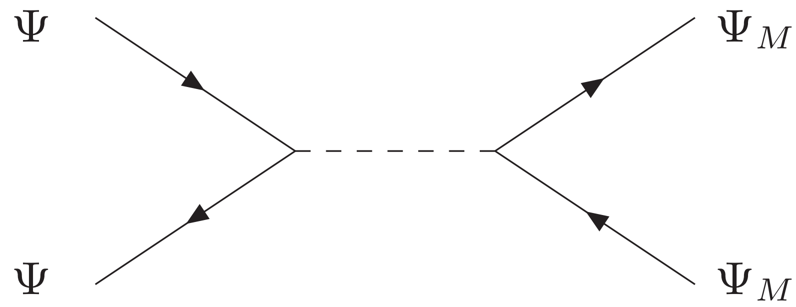



The amplitude is evaluated by following the arrows in reverse. Using:

$$\bar{v}(\vec{\boldsymbol{p}}_2, s_2)\Gamma v(\vec{\boldsymbol{p}}_4, s_4) = -\eta_\Gamma \bar{u}(\vec{\boldsymbol{p}}_4, s_4)\Gamma u(\vec{\boldsymbol{p}}_2, s_2)\,,$$

one can check that the invariant amplitudes resulting from diagram sets (i) and (ii) differ by an overall minus sign, as expected due to the fact that the corresponding order of the spinor wave functions differs by an odd permutation [e.g., for the $t$-channel graphs, compare 3142 and 3124 for (i) and (ii) respectively]. For the same reason, there is a relative minus sign between the $t$-channel and $u$-channel graphs for either diagram set [e.g., compare 3142 and 4132 in diagram set(i)].

If $s$-channel annihilation contributes, its calculation is straightforward.




Relative to the $t$-channel graph of diagram set (ii), this diagram comes with an extra minus sign [since 2134 is odd with respect to 3124].

In the computation of the unpolarized cross-section, non-standard spin projection operators can arise in the evaluation of the interference terms, e.g.¶

$$\sum_s u(\vec{\boldsymbol{p}}, s) v^T(\vec{\boldsymbol{p}}, s) = (\not{p} + m) C^T \,, \qquad \sum_s \bar{u}^T(\vec{\boldsymbol{p}}, s) \bar{v}(\vec{\boldsymbol{p}}, s) = C^{-1}(\not{p} - m) \,,$$

which requires additional manipulation of the charge conjugation matrix $C$. However, these non-standard spin projection operators can be avoided by judicious use of spinor wave function product relations of the kind displayed on the previous two pages.

¶see Appendix D of G.L. Kane and H.E. Haber, *Phys. Rep.* **117** (1985) 75.